Autour des équations d'Einstein dans le vide avec un champ de Killing spatial de translation.

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Autour des équations d'Einstein dans le vide avec un champ de Killing spatial de translation

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par

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Résumé

Dans cette thèse, nous étudions les équations d'Einstein dans le vide en dimension $3+1$, sur des variétés de la forme $\Sigma \times \mathbb{R}_{x_3} \times \mathbb{R}$, où Σ est une variété de dimension 2, munies d'une métrique de la forme

$$\mathbf{g} = e^{-2\gamma} g + e^{2\gamma} (dx^3)^2,$$

où x^3 est la coordonnée selon $\mathbf{R}_{x_3}$, γ est une fonction scalaire, et g une métrique Lorentzienne sur $\Sigma \times \mathbb{R}$, telles que ∂_{x_3} est un champ de vecteur de Killing. Pour ces métriques, les équations d'Einstein dans le vide sont équivalentes au système en dimension $2+1$

$$\begin{cases} \Box_g \gamma = 0 \\ R_{\mu\nu} = 2\partial_\mu \gamma \partial_\nu \gamma, \end{cases} \tag{0.0.1}$$

où $R_{\mu\nu}$ est le tenseur de Ricci associé à g.

La partie principale de cette thèse concerne le cas où $\Sigma = \mathbb{R}^2$. Dans ce cas, il existe une solution triviale, donnée par $\gamma = 0$, et g la métrique de Minkowski sur $\mathbb{R}^2 \times \mathbb{R}$. La question est alors d'étudier la stabilité non linéaire de cette solution triviale.

Les données initiales pour les équations d'Einstein ne peuvent pas être choisies de manière arbitraire, elles doivent satisfaire des relations de compatibilité appelées équations de contraintes. Nous étudions ces équations pour le système (0.0.1) dans la première partie de cette thèse. Le but est de trouver des solutions, $(\bar{g}, K)$, des équations de contraintes, qui soient asymptotiquement plates sur $\mathbb{R}^2$. Cependant la notion d'asymptotiquement plat n'est pas canonique en dimension 2. Par exemple, les ondes d'Einstein-Rosen, qui sont des solutions radiales du problème en $2+1$, possèdent un angle de défaut à l'infini spatial. En particulier, ces solutions ne convergent pas vers la métrique euclidienne à l'infini spatial. De plus, à cause du comportement de l'opérateur de Laplace sur $\mathbb{R}^2$, trouver des solutions aux équations de contraintes demande une analyse très particulière.

Les solutions que l'on construit dans cette thèse ont un comportement non trivial à l'infini. Leur développement asymptotique fait apparaître des quantités que l'on peut relier aux charges globales (comme la masse ADM, le moment ADM...)

Dans la seconde partie, nous prouvons la stabilité en temps exponentiel de la solution triviale. En s'inspirant de [40], nous aimerions travailler en coordonnées d'onde. Le système (0.0.1) s'écrit alors sous la forme

$$\begin{cases} \Box_g \gamma = 0 \\ \Box_g g_{\mu\nu} = -4\partial_\mu \gamma \partial_\nu \gamma + P_{\mu\nu}(g)(\partial g, \partial g), \end{cases} \tag{0.0.2}$$

où $P_{\mu\nu}$ est une forme quadratique. En dimension 2, le taux de dispersion des ondes libres, qui est seulement de $\frac{1}{\sqrt{t}}$, rend l'étude du système (0.0.2) assez difficile. Si nous regardons le problème modèle

$$\begin{cases} \Box \gamma = 0 \\ \Box h = (\partial \gamma)^2, \end{cases}$$

le taux de décroissance de γ ne donne aucune décroissance pour h. Pour avoir plus d'informations, nous allons adapter l'analyse des ondes d'Einstein-Rosen. Cela nous amène à introduire la famille de métriques suivante

$$g_b = -dt^2 + dr^2 + (r + \chi(q)b(\theta)q)^2 d\theta^2 + J(\theta)\chi(q)dqd\theta,$$

où $q = r - t$ et χ est une fonction cut-off telle que $\chi(q) = 1$ pour $q \geq 2$ et $\chi(q) = 0$ pour $q \leq 1$. Ces métriques sont Ricci plates pour $q \geq 2$. Afin de converger vers la solution de Minkowski à l'infini temporel, nous forçons la condition suivante à être réalisée

$$b(\theta) \sim \int_0^\infty (\partial\gamma)^2(t, r, \theta) r dr,$$

quand t tend vers $+\infty$.

Dans la dernière partie de cette thèse, nous étudions les équations de contraintes avec un champ de Killing de translation spatiale dans le cas hyperbolique compact, sans condition de courbure moyenne constante. Nous montrons l'existence d'une équation limite associée aux équations de contraintes, de la même manière que dans [16]. Ceci est un travail en collaboration avec Romain Gicquaud.

Abstract

In this thesis, we study solutions of the $3+1$ vacuum Einstein equations, on manifolds of the form $\Sigma \times \mathbb{R}_{x_3} \times \mathbb{R}$, where Σ is a 2 dimensional manifold, equipped with a metric of the form

$$\mathbf{g} = e^{-2\gamma} g + e^{2\gamma}(dx^3)^2,$$

where x^3 is the coordinate on $\mathbf{R}_{x_3}$, γ a scalar function, and g a Lorentzian metric on $\Sigma \times \mathbb{R}$, such that ∂_{x_3} is a Killing vector field. For these metrics, Einstein vacuum equations are equivalent to the $2+1$ dimensional system

$$\begin{cases} \Box_g \gamma = 0 \\ R_{\mu\nu} = 2\partial_\mu \gamma \partial_\nu \gamma, \end{cases} \tag{0.0.3}$$

where $R_{\mu\nu}$ is the Ricci tensor associated to g.

The main part of this thesis is concerned with the case where $\Sigma = \mathbb{R}^2$. In that case, there is a trivial solution, given by $\gamma = 0$, and g the Minkowski metric on $\mathbb{R}^2 \times \mathbb{R}$. The question is to study the nonlinear stability of this trivial solution.

The initial data for Einstein equations can not be chosen arbitrarily, they have to satisfy compatibility conditions known as the constraint equations. We study them for (0.0.3) in the first part of this thesis. The aim is to find asymptotically flat solutions $(\bar{g}, K)$ to the constraint equations in $\mathbb{R}^2$. However, the definition of an asymptotically flat manifold is not so clear in two dimensions. Einstein-Rosen waves are radial solutions of the $2+1$ dimensional problem with an angle at space-like infinity. In particular, these solutions do not tend to the Euclidean metric at space-like infinity. Moreover, the behaviour of the Laplace operator on $\mathbb{R}^2$ makes the problem of finding solutions to the constraint equations quite intricate.

The solutions we find have a non trivial behaviour at infinity. The asymptotic development of our solutions let appear quantities which seem to be the two dimensional equivalents of the global charges (ADM mass, ADM momentum,...).

In the second part, we prove the stability in exponential time of the trivial solution. Following [40], we would like to work in wave coordinates. Then, our system (0.0.3) takes the form

$$\begin{cases} \Box_g \gamma = 0 \\ \Box_g g_{\mu\nu} = -4\partial_\mu \gamma \partial_\nu \gamma + P_{\mu\nu}(g)(\partial g, \partial g), \end{cases} \tag{0.0.4}$$

where $P_{\mu\nu}$ is a quadratic form. In two dimensions, the decay of the free wave, which is only $\frac{1}{\sqrt{t}}$, makes the studying of (0.0.4) quite difficult. If we look at the model problem

$$\begin{cases} \Box \gamma = 0 \\ \Box h = (\partial \gamma)^2, \end{cases}$$

the decay of γ gives no decay at all for h. To obtain more informations, we will adapt the analysis of the Einstein-Rosen waves. This leads us to introduce a non trivial family of background metrics

$$g_b = -dt^2 + dr^2 + (r + \chi(q)b(\theta)q)^2 d\theta^2 + J(\theta)\chi(q)dq d\theta,$$

where $q = r - t$ and χ a cut-off function such that $\chi(q) = 1$ for $q \geq 2$ and $\chi(q) = 0$ for $q \leq 1$. These metrics are Ricci flat for $q \geq 2$. To have convergence at time-like infinity to the Minkowski solution, we have to enforce

$$b(\theta) \sim \int_0^\infty (\partial\gamma)^2(t, r, \theta) r dr,$$

as t tend to ∞.

In the last part of this thesis we study the constraint equations with a space-like Killing field without constant mean curvature assumption in the compact hyperbolic case. We show the existence of a limit equation associated to the constraint equations, as in [16].This is a joint work with Romain Gicquaud.

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Chapitre 1

Introduction

Dans la théorie de la relativité générale, l'espace-temps est décrit par un couple $(M, \mathbf{g})$ où M est une variété, généralement de dimension $3+1$, et $\mathbf{g}$ est une métrique lorentzienne, c'est à dire une métrique de signature $(-,+,+,+)$. Cette métrique doit satisfaire le système d'équations suivant, introduit par Einstein en 1915, et qui relie la courbure de l'espace-temps aux sources de matière et d'énergie présentes dans l'univers :

$$\mathbf{R}_{\alpha\beta} - \frac{1}{2}\mathbf{R}\mathbf{g}_{\alpha\beta} = \mathbf{T}_{\alpha\beta}. \tag{1.0.1}$$

Ici $\mathbf{R}_{\alpha\beta}$ désigne le tenseur de Ricci de $\mathbf{g}$, $\mathbf{R}$ la courbure scalaire de $\mathbf{g}$, et $\mathbf{T}$ le tenseur énergie-impulsion. Les objets géométriques $\mathbf{R}_{\alpha\beta}$ et $\mathbf{R}$ seront explicités dans la Section 1.1.1. Les distributions de matière et les champs autres que gravitationnels sont décrits par le tenseur énergie-impulsion $\mathbf{T}_{\alpha\beta}$. Voici quelques exemples :

- dans le vide, $\mathbf{T}_{\alpha\beta} = 0$, en prenant la trace de (1.0.1), on a alors $\mathbf{R} = 0$, et les équations (1.0.1) sont équivalentes à $\mathbf{R}_{\alpha\beta} = 0$.
- en présence d'un champ électro-magnétique
$$\mathbf{T}_{\alpha\beta} = \frac{1}{4\pi}\left(F^{\mu}_{\alpha}F_{\beta\mu} - \frac{1}{4}g_{\alpha\beta}F_{\mu\nu}F^{\mu\nu}\right),$$
où $F_{\alpha\beta}$ est le tenseur électro-magnétique,
- en présence d'un fluide parfait (par exemple pour modéliser l'intérieur d'une étoile)
$$\mathbf{T}_{\alpha\beta} = (\rho + p)u_\alpha u_\beta + p g_{\alpha\beta},$$
où u^α est le quadri-vecteur vitesse, p est la pression, et ρ est la densité d'énergie du fluide.

Dans cette thèse nous nous intéresserons au cas $\mathbf{T}_{\alpha\beta} = 0$.

Remark 1.0.1. *Les équations* (1.0.1) *relient des tenseurs, qui sont des objets indépendants d'un choix de repère. Cela s'appuie sur le* ***principe de covariance****. Ce principe postule que les lois de la physique ne dépendent pas du référentiel dans lequel on les exprime.*

Le plan du reste de ce chapitre est le suivant. Dans la section 1.1 nous introduisons les objets géométriques utiles à la compréhension de (1.0.1). Dans la section 1.2 nous présentons les équations de contraintes, qui seront abordées aux chapitres 2 et 4. Dans la section 1.3, nous présentons le problème de Cauchy pour les équations d'Einstein. La section 1.4 est une introduction aux problèmes d'existence globale à petites données initiales pour les équations d'onde non linéaires, qui est un domaine dans lequel s'inscrit le Chapitre 3. Dans la section 1.5, nous écrivons les équations d'Einstein dans le vide en présence d'un champ de Killing conforme de translation spatiale, qui est la symétrie qui nous intéressera tout au long de cette thèse. Enfin, dans la section 1.6, nous présentons les résultats démontrés dans cette thèse.

1.1 Préliminaires géométriques

Dans cette section nous rappelons quelques notions de géométrie riemannienne et lorentzienne bien connues. Nous nous appuyons sur [20] et [47].

1.1.1 Variétés riemanniennes, variétés lorentziennes

1.1.1.1 Métrique

En tout point x d'une variété M de dimension n, on peut définir l'espace tangent en x à M, noté T_xM. Le fibré tangent, noté TM, est l'union disjointe des fibres T_xM. Une section X du fibré tangent, aussi appelée champ de vecteurs, est une application lisse qui à tout x dans M associe un élément de T_xM. On note alors $X \in \Gamma(TM)$.

Pour $x \in M$, on peut aussi considérer l'espace des formes linéaires sur T_xM, $(T_xM)^*$. L'union des $(T_xM)^*$ est appelé fibré cotangent et est noté T^*M. A partir de TM et T^*M on peut considérer l'espace fibré

$$\underbrace{TM \otimes .. \otimes TM}_{\text{p fois}} \otimes \underbrace{T^*M \otimes .. \otimes T^*M}_{\text{q fois}}$$

pour p et q entiers naturels. Une section de ce fibré est un tenseur de type (p, q).

Définition 1.1.1. *Une métrique riemannienne sur M est la donnée d'un tenseur de type $(0, 2)$ symétrique défini positif. Une métrique lorentzienne sur M est la donnée d'une tenseur de type $(0, 2)$ symétrique et de signature $(1, n-1)$.*

1.1.1.2 Structure causale en géométrie lorentzienne

Soit M une variété munie d'une métrique lorentzienne $\mathbf{g}$. On note

$$\langle X, Y\rangle = \mathbf{g}_{\alpha\beta} X^\alpha Y^\beta.$$

Définition 1.1.2. *Soit X un champ de vecteur.*

1. *Si $\langle X, X\rangle < 0$ on dit que X est de genre temps.*
2. *Si $\langle X, X\rangle = 0$ on dit que X est nul.*
3. *Si $\langle X, X\rangle > 0$ on dit que X est de genre espace.*

On dira qu'une courbe est de genre temps si en tout point son vecteur tangent est de genre temps et qu'elle est causale si en tout point son vecteur tangent est soit de genre temps soit nul. On dira qu'une sous variété est de genre espace si tous ses vecteurs tangents sont de genre espace. Dans ce cas là, $\mathbf{g}$ restreinte à cette sous variété est riemannienne. On appelle cône de lumière l'ensemble des vecteurs nuls en un point.

1.1.1.3 Connexion

Soit M une variété différentielle et $M \to E$ un fibré vectoriel (E peut être par exemple le fibré tangent TM ou le fibré cotangent T^*M). Soit σ une section de $M \to E$, notée $\sigma \in \Gamma(E)$. Étant donné un champ de vecteur X sur M, une dérivée covariante ou connexion est une façon de définir la dérivée de σ dans la direction de X, de manière tensorielle par rapport à X, c'est à dire C^∞ linéairement par rapport à X.

Définition 1.1.3. *Une dérivée covariante ou connexion ∇ sur le fibré vectoriel $M \to E$ est une application*

$$\nabla \ : \ \Gamma(TM) \times \Gamma(E) \to \Gamma(E)$$

qui est C^∞ linéaire par rapport à la première variable et $\mathbb{R}$ linéaire par rapport à la seconde, et qui vérifie la règle de Leibniz suivante

$$\forall f \in C^\infty(M)\ \forall \sigma \in \Gamma(E), \quad \nabla_X(f\sigma) = df(X)\sigma + f\nabla_X\sigma.$$

Soit maintenant M une variété munie d'une métrique g riemannienne ou lorentzienne. Il existe alors une unique connexion satisfaisant la définition suivante :

Définition 1.1.4. *La connexion de Levi-Civita D est l'unique connexion sur TM telle que*

1. *D est métrique, c'est à dire $\forall X, Y, Z \in \Gamma(TM)$, $X(g(Y, Z)) = g(D_X Y, Z) + g(Y, D_X Z)$ où on a noté, pour une fonction scalaire f, $X(f) = df(X)$,*
2. *D est sans torsion, c'est à dire $\forall X, Y \in \Gamma(TM)$, $D_X Y - D_Y X = [X, Y]$, où $[.,.]$ est le crochet de Lie.*

En coordonnées, la connexion de Levi-Civita D associée à g s'écrit

$$D_X Y = X^i \frac{\partial Y^k}{\partial x^i}\partial_k + X^i \Gamma^j_{ik} Y^k \partial_j,$$

où les Γ^j_{ik} sont les symboles de Christoffel

$$\Gamma^j_{ik} = \frac{1}{2} g^{jl}(\partial_i g_{kl} + \partial_k g_{il} - \partial_l g_{ik}). \tag{1.1.1}$$

Une fois définie la connexion de Levi-Civita sur TM, on peut la définir sur tous les fibrés vectoriels associés à TM par compatibilité. Par exemple, pour $\sigma \in T^*M$, on a

$$X(\sigma(Y)) = (D_X\sigma)(Y) + \sigma(D_X Y),$$

ce qui permet de définir $D_X\sigma$.

1.1.2 Courbure

1.1.2.1 Définitions

Les dérivations usuelles satisfont la règle de Schwarz

$$\frac{\partial^2 f}{\partial x^i \partial x^j} = \frac{\partial^2 f}{\partial x^j \partial x^i},$$

où f est une fonction scalaire. Ceci est faux en général pour une connexion :

$$\nabla_{\frac{\partial}{\partial x^i}} \nabla_{\frac{\partial}{\partial x^j}} \sigma \neq \nabla_{\frac{\partial}{\partial x^j}} \nabla_{\frac{\partial}{\partial x^i}} \sigma,$$

pour σ un tenseur. Le tenseur de courbure est l'objet qui mesure ce défaut de commutativité.

Définition 1.1.5. *Soit M une variété munie d'une métrique g. Le tenseur de courbure associé à la connexion de Levi-Civita D est défini par*

$$\forall X, Y, Z \in \Gamma(TM),\ R(X, Y)Z = D_X D_Y Z - D_X D_Y Z - D_{[X,Y]}Z.$$

En coordonées

$$R_{\alpha\beta}{}^{\lambda}{}_{\mu} v^\mu = (D_\alpha D_\beta - D_\beta D_\alpha)v^\lambda.$$

L'expression $R(X, Y)Z$ est C^∞ linéaire par rapport à X, Y et Z, et est antisymétrique en X et Y. On appelle aussi ce tenseur le tenseur de Riemann. En coordonnées, il s'exprime de la manière suivante

$$R_{\alpha\beta}{}^{\lambda}{}_{\mu} = \partial_\alpha \Gamma^\lambda_{\beta\mu} - \partial_\beta \Gamma^\lambda_{\alpha\mu} + \Gamma^\lambda_{\alpha\rho}\Gamma^\rho_{\beta\mu} - \Gamma^\lambda_{\beta\rho}\Gamma^\rho_{\alpha\mu}. \tag{1.1.2}$$

Définition 1.1.6. *Le tenseur de Ricci est donné par la trace du tenseur de Riemann*

$$R_{\alpha\beta} = R_{\lambda\alpha}{}^{\lambda}{}_{\beta}.$$

Definition 1.1.7. *La courbure scalaire est la trace du tenseur de Ricci dans la métrique g :*

$$R = g^{\alpha\beta} R_{\alpha\beta}.$$

1.1.2.2 Propriétés du tenseur de Riemann

Proposition 1.1.8. *Le tenseur de Riemann vérifie les propriétés suivantes :*

1. $R_{ijkl} = -R_{jikl}$,
2. $R_{ijkl} = R_{klij}$,
3. *Première identité de Bianchi :* $R_{ijk}{}^l + R_{jki}{}^l + R_{kij}{}^l = 0$,
4. *Deuxième identité de Bianchi :* $D_i R_{jkl}{}^m + D_j R_{kil}{}^m + D_k R_{ijl}{}^m$.

Si on note G_{ij} le tenseur d'Einstein

$$G_{ij} = R_{ij} - \frac{1}{2} R g_{ij},$$

on obtient l'équation fondamentale suivante

$$D^i G_{ij} = 0, \tag{1.1.3}$$

en contractant deux fois la deuxième identité de Bianchi.

1.1.3 Dérivée de Lie, champs de Killing et seconde forme fondamentale

1.1.3.1 Dérivée de Lie

La dérivée de Lie est une notion de dérivation différente de la connexion. Pour la définir, on a besoin d'introduire la notion de tiré en arrière ou pull-back. A partir d'un difféomorphisme φ, on peut tirer en arrière des tenseurs. Si $\varphi : U \to V$ est un difféomorphisme entre deux ouverts d'une variété M, et si ω est une forme linéaire définie sur V, on peut définir sur U le tiré en arrière de ω, noté $\varphi^* \omega$ par

$$\forall x \in U,\ \forall X \in T_x M,\ \varphi^* \omega_x(X) = \omega_{\varphi(x)}(d\varphi(x) X)$$

où $d\varphi(x)$ est la différentielle de φ au point x.

Le tiré en arrière d'un champ de vecteur Y sur V est défini par

$$\varphi^* Y(x) = (d\varphi(x))^{-1} Y(\varphi(x)).$$

Le tiré en arrière d'un tenseur quelconque se définit naturellement à partir des deux définitions précédentes.

Soit X un champ de vecteur sur une variété M. On note $\varphi : I \times M \to M$, où I est un intervalle de $\mathbb{R}$, le flot associé à X, c'est à dire la solution de l'équation différentielle

$$\frac{d}{dt} \varphi_t(x) = X(\varphi_t(x)),$$
$$\varphi_0(x) = x.$$

Pour tout t, φ_t est un difféomorphisme local là où il est défini.

Définition 1.1.9. *La dérivée de Lie d'un tenseur T dans la direction X est*

$$\mathcal{L}_X T = \frac{d}{dt} (\varphi^*_{-t} T)|_{t=0}.$$

Remarque 1.1.10.

1. *La dérivée de Lie ainsi définie satisfait la règle de Leibniz.*
2. *Contrairement à la dérivée covariante, $\mathcal{L}_X$ n'est pas C^∞ linéaire par rapport à X.*

3. *Les trois notions de dérivation coïncident sur les fonctions scalaires :*

$$\mathcal{L}_X f = D_X f = df(X).$$

Proposition 1.1.11. *La dérivée de Lie selon X d'un champ de vecteur Y correspond au crochet de Lie*

$$\mathcal{L}_X Y = [X, Y].$$

De plus, si g est une métrique et D sa connexion de Levi-Civita associée, on a

$$(\mathcal{L}_X g)_{ab} = D_a X_b + D_b X_a.$$

1.1.3.2 Champs de Killing

Définition 1.1.12. *Soit M une variété munie d'une métrique g. Un champ de vecteur X est un champ de Killing si son flot est un flot d'isométries, ou de manière équivalente si*

$$\mathcal{L}_X g = 0.$$

On dit que X est un champ de vecteurs Killing conforme si

$$\mathcal{L}_X g = \Omega g,$$

où Ω est une fonction scalaire.

Exemple 1.1.13. *Dans l'espace-temps de Minkowski $(\mathbb{R}^{1+n}, m)$, les champs de Killing sont*

- *Les translations :* $T_\alpha = \frac{\partial}{\partial x_\alpha}$,
- *Les rotations :* $\Omega_{ij} = x_i \frac{\partial}{\partial x_j} - x_j \frac{\partial}{\partial x_i}$,
- *Les rotations hyperboliques :* $\Omega_{0i} = t \frac{\partial}{\partial x_i} + x_i \frac{\partial}{\partial t}$.

Le changement d'échelle, correspond par contre à un champ de Killing conforme :

$$S = t\frac{\partial}{\partial t} + x_i \frac{\partial}{\partial x_i},$$

est tel que

$$\mathcal{L}_S m = 2m.$$

1.1.3.3 Seconde forme fondamentale

Soit M une variété munie d'une métrique g, riemannienne ou lorentzienne. On notera $g(X,Y) = \langle X, Y\rangle$. Soit Σ une hypersurface et T un champ de vecteur orthogonal à Σ tel que $\langle T, T\rangle$ vaille 1 ou -1. Soit D la connexion de Levi-Civita associée à g et $\bar{D}$ la connexion induite sur Σ.

Définition 1.1.14. *La seconde forme fondamentale, K est définie par*

$$D_X Y = \bar{D}_X Y - K(X,Y)T.$$

On a alors si $\langle T, T\rangle = 1$,

$$K(X,Y) = \langle D_X T, Y\rangle,$$

et si $\langle T, T\rangle = -1$,

$$K(X,Y) = -\langle D_X T, Y\rangle.$$

La propriété suivante donne une définition plus intuitive de la seconde forme fondamentale :

Proposition 1.1.15. *Soit $\bar{g}$ la métrique induite sur Σ. Alors $K = \frac{1}{2}\left(\mathcal{L}_T g\right)|_{T^*\Sigma \times T^*\Sigma}$ si $\langle T, T\rangle = 1$ et $K = -\frac{1}{2}\left(\mathcal{L}_T g\right)|_{T^*\Sigma \times T^*\Sigma}$ si $\langle T, T\rangle = -1$.*

1.1.4 Feuilletage de l'espace temps, lapse et shift

Pour mieux comprendre l'équation (1.0.1), il est important d'avoir une décomposition un peu plus explicite du tenseur de Ricci. Dans ce paragraphe, on suivra [8]. Soit $(M, \mathbf{g})$ une variété lorentzienne telle que $M = \Sigma \times \mathbb{R}$ où les $\Sigma_t = \Sigma \times \{t\}$ sont de genre espace. On choisit des coordonnées locales $(x^1, .., x^i, .., x^n, t)$ adaptées à la structure produit. On prend ensuite une base de champs de vecteurs (e_i, e_0) avec $e_i = \partial_i$ tangent à Σ_t et e_0 orthogonal à Σ_t à déterminer. Soit (θ^i, θ^0) la base duale. On impose

$$\theta^0 = dt.$$

On peut alors écrire

$$e_0 = \partial_t - \beta^j \partial_j.$$

On notera par analogie $e_0 = \partial_0$ même si cela ne correspond pas à une coordonnée. Le champ de vecteurs β^j est appelé le shift. θ^i s'écrit alors

$$\theta^i = dx^i + \beta^i dt.$$

Dans la base (θ^i, θ^0) la métrique s'écrit

$$\mathbf{g} = -N^2(\theta^0)^2 + g_{ij}\theta^i\theta^j.$$

où g est la métrique riemannienne induite par $\mathbf{g}$ sur Σ_t et

$$-N^2 = \mathbf{g}(e_0, e_0).$$

N est appelé le lapse. Notons

$$T = \frac{1}{N} e_0.$$

On a alors

$$\mathbf{g}(T, T) = -1$$

et on peut définir la seconde forme fondamentale

$$K_{ij} = -\langle \mathbf{D}_{e_i} T, e_j \rangle = -\frac{1}{N}\langle \mathbf{D}_{e_i} e_0, e_j \rangle = -\frac{1}{2N}(\mathcal{L}_{e_0} g_{ij}),$$

où l'on a noté $\mathcal{L}_{e_0}$ la dérivée de Lie dans la direction e_0 (voir la section 1.1.3).

Notons D la connexion associée à g et R son tenseur de courbure associé.

Proposition 1.1.16. *On a la décomposition suivante du tenseur de Ricci associé à* $\mathbf{g}$ *:*

$$\mathbf{R}_{ij} = R_{ij} + K_{ij}K_l{}^l - 2K_i{}^l K_{jl} - N^{-1}(\mathcal{L}_{e_0} K_{ij} + D_i\partial_j N), \tag{1.1.4}$$

$$\mathbf{R}_{0j} = N(\partial_j K^h{}_h - D_h K^h{}_j), \tag{1.1.5}$$

$$\mathbf{R}_{00} = N(\partial_0(K^h{}_h) - N K_{ij}K^{ij} + \Delta N), \tag{1.1.6}$$

ainsi que la formule suivante pour la courbure scalaire :

$$\mathbf{R} = R + (\operatorname{tr} K)^2 + K^{ij}K_{ij} - 2N^{-1}\partial_0(K^h{}_h) - 2N^{-1}\Delta N. \tag{1.1.7}$$

Démonstration. La preuve de la proposition est classique et nous la rappelons ci-dessous. Montrons d'abord la décomposition suivante du tenseur de Riemann :

$$\mathbf{R}_{ijkl} = R_{ijkl} - K_{kj}K_{il} + K_{ik}K_{jl}, \tag{1.1.8}$$

$$\mathbf{R}_{0ijk} = N(D_j K_{ki} - D_k K_{ij}), \tag{1.1.9}$$

$$\mathbf{R}_{0i0j} = N(\mathcal{L}_{e_0} K_{ij} + N K_{ik}K^k{}_j + D_i\partial_j N). \tag{1.1.10}$$

Prouvons d'abord (1.1.8), qui n'est autre que le théorème Egregium de Gauss. On utilise la commutation des e_i et la définition de la seconde forme fondamentale pour obtenir :

$$\begin{aligned}\mathbf{R}_{ijkl} &= \langle \mathbf{D}_i\mathbf{D}_j e_l - \mathbf{D}_j\mathbf{D}_i e_l, e_k\rangle \\ &= \langle \mathbf{D}_i(D_j e_l - K_{lj}T) - \mathbf{D}_j(D_i e_l - K_{il}T), e_k\rangle \\ &= R_{ijkl} - K_{lj}\langle \mathbf{D}_i T, e_k\rangle + K_{il}\langle D_j T, e_k\rangle \\ &= R_{ijkl} + K_{lj}K_{ik} - K_{il}K_{jk}.\end{aligned}$$

Prouvons (1.1.9). On a

$$\begin{aligned}\mathbf{R}_{0ijk} = -\mathbf{R}_{jki0} &= N\langle \mathbf{D}_j\mathbf{D}_k T - \mathbf{D}_k\mathbf{D}_j T, e_i\rangle \\ &= -N\big(D_j\langle \mathbf{D}_k T, e_i\rangle - \langle \mathbf{D}_k T, D_j e_i\rangle - D_k\langle \mathbf{D}_j T, e_i\rangle + \langle \mathbf{D}_j T, D_k e_i\rangle\big) \\ &= -N\big(-D_j(K(e_k,e_i)) + K(e_k, D_j e_i) + D_k(K(e_j,e_i)) - K(e_j, D_k e_i)\big) \\ &= -N\big(-D_j K_{ki} - K(e_i, D_j e_k) + D_k K_{ij} + K(e_i, D_k e_j)\big),\end{aligned}$$

où on a utilisé la définition de la dérivée covariante d'un tenseur. On a donc

$$\begin{aligned}\mathbf{R}_{0ijk} &= -N(D_k K_{ij} - D_j K_{ki} - K([e_j,e_k],e_i)) \\ &= -N(D_k K_{ij} - D_j K_{ki}),\end{aligned}$$

car les e_i commutent. Montrons maintenant (1.1.10). On a :

$$\mathbf{R}_{0i0j} = -N\langle \mathbf{D}_0\mathbf{D}_i T - \mathbf{D}_i\mathbf{D}_0 T - \mathbf{D}_{[e_0,e_i]}T, e_j\rangle.$$

Or

$$\begin{aligned}\langle \mathbf{D}_0 T, e_i\rangle &= -\langle T, \mathbf{D}_0 e_i\rangle \\ &= -\langle T, \mathbf{D}_i e_0 + [e_0,e_i]\rangle = -\langle T, (\partial_i N)T + N\mathbf{D}_i T + (\partial_i\beta^j)e_j\rangle = \partial_i N,\end{aligned}$$

où l'on a utilisé en particulier le fait que $e_0 = NT$ et $[\partial_i,\partial_j] = [\partial_i,\partial_t] = 0$. Donc $\mathbf{D}_0 T = \nabla N$ et

$$\begin{aligned}\mathbf{R}_{0i0j} &= -N(\partial_0\langle \mathbf{D}_i T, e_j\rangle - \langle \mathbf{D}_i T, \mathbf{D}_0 e_j\rangle - \langle D_i\nabla N, e_j\rangle - \langle \mathbf{D}_{[e_0,e_i]}T, e_j\rangle) \\ &= -N(\partial_0\langle \mathbf{D}_i T, e_j\rangle - \langle \mathbf{D}_i T, [e_0,e_j]\rangle + K_i{}^l\langle e_l, \mathbf{D}_j e_0\rangle - \langle D_i\nabla N, e_j\rangle - \langle \mathbf{D}_{[e_0,e_i]}T, e_j\rangle) \\ &= -N(-\partial_0(K(e_i,e_j)) + K(e_i, \mathcal{L}_{e_0}e_j) + K(\mathcal{L}_{e_0}e_i, e_j) - NK_i{}^lK_{lj} - D_i\partial_j N) \\ &= -N(-\mathcal{L}_{e_0}K_{ij} - NK_i{}^lK_{lj} - D_i\partial_j N).\end{aligned}$$

On peut maintenant prouver la décomposition de la proposition. En utilisant (1.1.8) et (1.1.10) on obtient pour (1.1.4) :

$$\begin{aligned}\mathbf{R}_{ij} &= \mathbf{g}^{\mu\nu}\mathbf{R}_{\mu i\nu j} \\ &= g^{lk}(R_{likj} + K_{ij}K_{lk} - K_{ik}K_{lj}) + \mathbf{g}^{00}N(\mathcal{L}_{e_0}K_{ij} + NK_i{}^lK_{lj} + D_i\partial_j N) \\ &= R_{ij} + K_{ij}K_l{}^l - K_i{}^lK_{jl} - N^{-1}(\mathcal{L}_{e_0}K_{ij} + NK_i{}^lK_{lj} + D_i\partial_j N).\end{aligned}$$

En utilisant (1.1.9) on obtient la preuve de (1.1.5) :

$$\mathbf{R}_{0j} = \mathbf{g}^{\mu\nu}\mathbf{R}_{\mu 0j\nu} = -g^{ik}\mathbf{R}_{0ijk} = N(D_j(g^{ik}K_{ik}) - D_k(g^{ik}K_{ij})).$$

En utilisant (1.1.10) on obtient la preuve de (1.1.6) :

$$\begin{aligned}\mathbf{R}_{00} &= \mathbf{g}^{\mu\nu}R_{\mu 00\nu} \\ &= -g^{ij}\mathbf{R}_{0i0j} = N(\partial_0(g^{ij}K_{ij}) - (\mathcal{L}_{e_0}g^{ij})K_{ij} + Ng^{ij}K_{ik}K^k{}_j + g^{ij}D_i\partial_j N) \\ &= N(\partial_0(K^i{}_i) - 2NK^{ij}K_{ij} + NK^{ij}K_{ij} + \Delta N).\end{aligned}$$

On a utilisé dans la dernière égalité le fait suivant

$$\mathcal{L}_{e_0} g^{ij} = 2NK^{ij}.$$

En effet

$$0 = \partial_0(g^{ij}g_{jk}) = g^{ij}\mathcal{L}_{e_0}g_{jk} + g_{jk}\mathcal{L}_{e_0}g^{ij} = -2Ng^{ij}K_{jk} + g_{jk}\mathcal{L}_{e_0}g^{ij}.$$

En utilisant (1.1.4), (1.1.6) et $\mathcal{L}_{e_0}g^{ij} = 2NK^{ij}$ on obtient la formule pour la courbure scalaire :

$$\begin{aligned}
\mathbf{R} =& \mathbf{g}^{\mu\nu}\mathbf{R}_{\mu\nu} \\
=& g^{ij}(R_{ij} + K_{ij}K^h{}_h - 2K_{ik}K^k{}_j - N^{-1}\mathcal{L}_{e_0}K_{ij} - N^{-1}D_i\partial_j N) \\
&+ \mathbf{g}^{00}N(\partial_0(K^h{}_h) - NK_{ij}K^{ij} + \Delta N) \\
=& R + (\operatorname{tr} K)^2 + K^{ij}K_{ij} - 2N^{-1}\partial_0(K^h{}_h) - 2N^{-1}\Delta N.
\end{aligned}$$

Ceci conclut la preuve de la proposition. □

1.2 Équations de contraintes

On considère les équations d'Einstein dans le vide

$$\mathbf{G}_{\alpha\beta} = \mathbf{R}_{\alpha\beta} - \frac{1}{2}\mathbf{g}_{\alpha\beta}\mathbf{R} = 0. \tag{1.2.1}$$

On se donne comme données initiales un triplet (Σ, g, K) où

- Σ est une variété de dimension 3,
- g est une métrique riemannienne sur Σ,
- K est un 2-tenseur symétrique sur Σ.

Résoudre les équations d'Einstein dans le vide avec données initiales (Σ, g, K) c'est trouver un couple $(M, \mathbf{g})$ où M est une variété de dimension $3+1$, $\mathbf{g}$ une métrique lorentzienne, tel que

- La métrique $\mathbf{g}$ satisfait les équations d'Einstein (1.2.1),
- Σ s'injecte dans M, avec $\mathbf{g}|_\Sigma = g$,
- K est la seconde forme fondamentale associée à l'inclusion $\Sigma \subset M$.

Grâce à (1.2.1), (1.1.5),(1.1.6) et (1.1.7) les composantes temporelles du tenseur d'Einstein s'écrivent

$$\mathbf{G}_{00} = \frac{N^2}{2}(R - K_{ij}K^{ij} + (K^h{}_h)^2),$$
$$\mathbf{G}_{0i} = N(\partial_j(K^h{}_h) - D_hK^h{}_j).$$

Les composantes temporelles de (1.2.1) s'écrivent donc

$$\partial_j(K^h{}_h) - D_hK^h{}_j = 0, \tag{1.2.2}$$
$$R - K_{ij}K^{ij} + (K^h{}_h)^2 = 0. \tag{1.2.3}$$

Ces équations ne dépendent que de la donnée de g et K sur Σ. Elles doivent donc être vérifiées par les données initiales. On les appelle équations de contraintes. Plus précisément, (1.2.2) est la contrainte de moment et (1.2.3) la contrainte hamiltonienne.

Les équations de contrainte sont en fait une condition nécessaire et suffisante pour qu'un triplet (Σ, g, K) puisse s'immerger dans un couple (M, g) solution des équations d'Einstein. Nous citons le théorème pionnier suivant du à Choquet-Bruhat et Geroch [9]

Théorème 1.2.1 ([9]). *Tout triplet (Σ, g, K) de données initiales suffisamment lisses satisfaisant les équations de contraintes admet un unique développement maximal globalement hyperbolique.*

Les équations de contraintes sont sous-déterminées. La méthode la plus efficace à l'heure actuelle pour les résoudre est la méthode conforme, que nous présentons dans la section suivante.

1.2.1 La méthode conforme

1.2.1.1 Transformation conforme

Dans ce paragraphe, nous montrons comment sont transformés les tenseurs de courbure sous l'action d'une transformation conforme. Soit g une métrique riemannienne ou lorentzienne sur une variété de dimension n.

Proposition 1.2.1. *Écrivons $\widetilde{g} = e^{2\varphi} g$. Alors le tenseur de Ricci $\widetilde{R}_{ij}$ dans la métrique $\widetilde{g}$ s'écrit*

$$\widetilde{R}_{ij} = R_{ij} - g_{ij} D^k \partial_k \varphi - (n-2) D_i \partial_j \varphi + (n-2)(\partial_i \varphi \partial_j \varphi - g_{ij} \partial^k \varphi \partial_k \varphi).$$

L'expression pour la courbure scalaire est

$$e^{2\varphi} \widetilde{R} - R = -2(n-1) D^k \partial_k \varphi - (n-2)(n-1) \partial^k \varphi \partial_k \varphi.$$

Démonstration. Commençons par calculer le symbole de Christoffel associé à $\widetilde{g}$ à partir de la relation (1.1.1)

$$\begin{aligned} \widetilde{\Gamma}^i_{jk} &= \Gamma^i_{jk} + \frac{1}{2} e^{-2\varphi} g^{il} (g_{lj} \partial_k e^{2\varphi} + g_{lk} \partial_j e^{2\varphi} - g_{jk} \partial_l e^{2\varphi}) \\ &= \Gamma^i_{jk} + S^i_{jk}. \end{aligned} \tag{1.2.4}$$

où l'on a noté

$$S^i_{jk} = \delta^i_j \partial_k \varphi + \delta^i_k \partial_j \varphi - g_{jk} \partial^i \varphi.$$

Le tenseur de Ricci peut ensuite se calculer à partir du symbole de Christoffel

$$\begin{aligned} \widetilde{R}_{ij} =& \partial_k \widetilde{\Gamma}^k_{ij} - \partial_i \widetilde{\Gamma}^k_{kj} + \widetilde{\Gamma}^k_{ij} \widetilde{\Gamma}^l_{lk} - \widetilde{\Gamma}^k_{li} \widetilde{\Gamma}^l_{kj} \\ =& R_{ij} + \partial_l S^l_{ij} - \partial_i S^l_{lj} + \Gamma^k_{ij} S^l_{lk} - \Gamma^k_{li} S^l_{kj} + S^k_{ij} \Gamma^l_{lk} - S^k_{li} \Gamma^l_{kj} + S^k_{ij} S^l_{lk} - S^k_{li} S^l_{kj} \\ =& R_{ij} - (n-2) \partial_i \partial_j \varphi - \partial_l (g_{ij} \partial^l \varphi) + n \Gamma^k_{ij} \partial_k \varphi - \Gamma^l_{li} \partial_j \varphi - \Gamma^k_{ij} \partial_k \varphi + \Gamma^k_{il} g_{kj} \partial^l \varphi \\ & + \partial_j \varphi \Gamma^l_{li} + \partial_i \varphi \Gamma^l_{lj} - g_{ij} \partial^k \varphi \Gamma^l_{lk} - \partial_i \varphi \Gamma^k_{kj} - \partial_l \varphi \Gamma^l_{ij} + g_{il} \partial^k \varphi \Gamma^l_{kj} \\ & + (\delta^k_i \partial_j \varphi + \delta^k_j \partial_i \varphi - g_{ij} \partial^k \varphi)(n \partial_k \varphi) \\ & - (\delta^k_l \partial_i \varphi + \delta^k_i \partial_l \varphi - g_{il} \partial^k \varphi)(\delta^l_j \partial_k \varphi + \delta^l_k \partial_j \varphi - g_{kj} \partial^l \varphi) \\ =& R_{ij} - (n-2)(\partial_i \partial_j \varphi - \Gamma^k_{ij} \partial_k \varphi) - g_{ij} (\partial_l \partial^l \varphi + \Gamma^l_{lk} \partial^k \varphi) \\ & + 2n \partial_i \varphi \partial_j \varphi - n g_{ij} \partial_l \varphi \partial^l \varphi - (n+2) \partial_i \varphi \partial_j \varphi + 2 g_{ij} \partial_l \varphi \partial^l \varphi \\ =& R_{ij} - g_{ij} D_l \partial^l \varphi - (n-2) D_i \partial_j \varphi + (n-2)(\partial_i \varphi \partial_j \varphi - g_{ij} \partial^l \varphi \partial_l \varphi). \end{aligned}$$

En prenant la trace on obtient l'expression pour la courbure scalaire. □

1.2.1.2 Écriture des équations de contraintes

On se donne Σ une variété de dimension 3. Pour lever l'indétermination des équations (1.2.2) et (1.2.3), la méthode conforme, introduite par Lichnerowicz [37] et Choquet-Bruhat et York [10] consiste à fixer une métrique riemannienne $\widetilde{g}$ sur Σ, ainsi qu'un 2 tenseur symétrique σ, sans trace et sans divergence, et un champ scalaire τ, puis à chercher les solutions de (1.2.2) et (1.2.3) sous la forme

$$g = \varphi^4 \widetilde{g}, \tag{1.2.5}$$

$$K = \varphi^{-2}(\sigma + LW) + \frac{1}{3}\varphi^4 \widetilde{g}\tau, \tag{1.2.6}$$

où φ est un champ scalaire, W un champ de vecteurs et L l'opérateur Killing conforme

$$(LW)_{ij} = D_i W_j + D_j W_i - \frac{2}{3}\widetilde{g}_{ij} D^l W_l. \tag{1.2.7}$$

Grâce à la Proposition 1.2.1, les inconnues φ et W sont solutions du système suivant (voir le chapitre 7 de [8])

$$\begin{cases} div_{\widetilde{g}} LW = 2\varphi^6 d\tau, & (1.2.8a) \\ 8\Delta_{\widetilde{g}}\varphi - R_{\widetilde{g}}\varphi = \frac{2}{3}\tau^2\varphi^5 - |\sigma + LW|^2_{\widetilde{g}}\varphi^{-7}, & (1.2.8b) \end{cases}$$

où $\Delta_{\widetilde{g}}$ est l'opérateur de Laplace-Beltrami

$$\Delta_{\widetilde{g}} = D^\alpha D_\alpha u.$$

Ce système est de type elliptique. La manière la plus intuitive de le résoudre est de choisir la courbure moyenne τ constante sur Σ. Le système (1.2.8) n'est alors plus couplé et la difficulté réside dans la résolution de la deuxième équation, non linéaire, appelée équation de Lichnerowicz. Ce choix est appelé CMC (pour "constant mean curvature"). Dans les sections suivantes, nous citerons quelques résultats, pour Σ compacte puis pour Σ asymptotiquement plate, en suivant l'article de revue [14].

1.2.2 Le cas compact

On suppose Σ compact. L'existence et l'unicité de solutions à (1.2.8) ont été étudiées de manière exhaustive par divers auteurs, et sont reliées très fortement au problème de Yamabe, qui consiste à chercher des métriques de courbure scalaire constante dans la classe conforme d'une métrique donnée. On introduit l'invariant de Yamabe

$$\mathcal{Y}([g]) = \inf_{f \in C^\infty(\Sigma), f \not\equiv 0} \frac{\int_\Sigma (|\nabla f|^2 + \frac{1}{12}R(g)f^2)d\mu_g}{\|f\|^2_{L^6}}.$$

$\mathcal{Y}([g])$ est un invariant conforme, c'est à dire qu'il ne dépend que de la classe conforme de g. Le théorème suivant classifie les cas où une solution existe (voir [30])

Théorème 1.2.2 ([30])**.** *Soient $(\widetilde{g}, \sigma, \tau)$ des données C^∞, avec τ constant. Alors il existe toujours une solution W de* (1.2.8a)*. De plus, il existe une unique solution $\varphi > 0$ de l'équation de Lichnerowicz, si et seulement si on est dans l'un des cas suivants*

- $\mathcal{Y}([\widetilde{g}]) > 0$, $LW + \sigma \not\equiv 0$,
- $\mathcal{Y}([\widetilde{g}]) = 0$, $LW + \sigma \not\equiv 0$, $\tau \neq 0$,
- $\mathcal{Y}([\widetilde{g}]) < 0$, $\tau \neq 0$,
- $\mathcal{Y}([\widetilde{g}]) = 0$, $LW + \sigma \equiv 0$, $\tau = 0$.

Des résultats ont aussi été montrés pour dans le cas "presque CMC", c'est à dire avec $\frac{d\tau}{\tau}$ petit, par divers auteurs. Plus récemment, des résultats loin de "CMC" ont été obtenus ([27], [42], [16]). Nous donnons ici quelques détails sur le théorème principal de [16]

Théorème 1.2.3 ([16]). *Soit $(\Sigma, \widetilde{g})$ une variété riemannienne compacte qui ne possède pas de champs de Killing conforme. On suppose que τ ne s'annule nul part. Sous de bonnes conditions de régularité sur $\widetilde{g}, \tau, \sigma$, au moins l'une des assertions suivantes est vraie*

- *Il existe une solution (φ, W) de* (1.2.8) *avec $\varphi > 0$,*
- *Il existe une solution non triviale W de*

$$div(LW) = \alpha_0 \sqrt{\frac{2}{3}} |LW| \frac{d\tau}{\tau}. \tag{1.2.9}$$

L'équation (1.2.9) *est appelée équation limite.*

Un résultat similaire en présence d'un champ de Killing de translation spatiale fait l'objet du Chapitre 4.

1.2.3 Le cas asymptotiquement plat

Des données initiales (Σ, g, K) sont asymptotiquement plates si il existe C compact, $C \subset \Sigma$, tel que

$$\Sigma \setminus C = \cup E_n$$

où l'ensemble des E_n est fini, les E_n sont disjoints deux à deux et difféomorphes à $\mathbb{R}^3 \setminus B(0,1)$. De plus sur chaque E_n on suppose pour un certain $\rho > 0$

$$|\partial_x^\alpha (g - \delta)| = O(|x|^{-\alpha-\rho}), \ |\partial_x^\alpha K| = O(|x|^{-1-\rho-|\alpha|}).$$

Sur une variété asymptotiquement plate, la condition CMC correspond à $\tau = 0$. De plus, on a alors $W = 0$. On introduit l'équivalent de l'invariant de Yamabe dans le cas asymptotiquement plat

$$\mathcal{Y}_{AF}([g]) = \inf_{f \in C_c^\infty(\Sigma), f \not\equiv 0} \frac{\int_\Sigma (|\nabla f|^2 + \frac{1}{12} R(g) f^2) d\mu_g}{\|f\|_{L^6}^2}.$$

Le théorème suivant est du à Brill et Cantor.

Théorème 1.2.4 ([7]). *On suppose $(M, \widetilde{g})$ asymptotiquement plate, avec $\widetilde{g}, \sigma$ suffisamment régulières et décroissantes. Il existe une solution $\varphi > 0$ de l'équation de Lichnerowicz si et seulement si $\mathcal{Y}_{AF}([\widetilde{g}]) > 0$.*

Des résultats dans le cas presque CMC ont été montrés dans [12].

1.3 Problème de Cauchy en relativité générale

1.3.1 Les coordonnées d'onde

Soit $\mathbf{g}$ une métrique sur $\Sigma \times \mathbb{R}$. Si les équations de contraintes sont satisfaites sur $\Sigma \times \{0\}$, et si les composantes spatiales de (1.2.1) sont satisfaites partout, i.e. $G_{ij} = 0$, alors les contraintes sont satisfaites partout. En effet, les identités de Bianchi contractées (1.1.3) et le fait que $G_{ij} = 0$ impliquent

$$\mathbf{D}^\alpha \mathbf{G}_{\alpha 0} = 0, \tag{1.3.1}$$

$$\mathbf{D}^0 \mathbf{G}_{i0} = 0. \tag{1.3.2}$$

Comme $\mathbf{g}$ est connue, ceci est un système linéaire du premier ordre à quatre équations et quatre inconnues (qui sont les $\mathbf{G}_{0i}$) dont la solution est unique. Si les $\mathbf{G}_{0i}$ sont nuls à l'instant initial, ils sont donc nuls partout.

Plaçons nous pour simplifier en dimension $3+1$. Le système (1.2.1) est à priori un système de 10 équations à 10 inconnues (par symétrie). Cependant, parmi ces équations, il y a 4 équations de contraintes. Le système est donc sous déterminé. Cela correspond au fait que les équations d'Einstein sont invariantes par difféomorphisme et que deux espaces difféomorphes représentent physiquement le même espace-temps. Le choix d'un difféomorphisme correspond à 4 paramètres. Ce choix est appelé choix de jauge. Voici quelques exemples de choix de jauge :

- Les coordonnées d'ondes : ce sont des coordonnée x^μ telles que $\mathbf{D}_\alpha \mathbf{D}^\alpha x^\mu = 0$,
- Les coordonnées d'ondes généralisées : ce sont des coordonnée x^μ telles que $\mathbf{D}_\alpha \mathbf{D}^\alpha x^\mu = F^\mu$, où F^μ est une expression qui peut ou non dépendre de $\mathbf{g}$,
- La jauge maximale : on impose $\tau = \beta = 0$.

Dans le paragraphe suivant, nous allons détailler la formulation des équations d'Einstein en coordonnées d'onde. On rappelle que les coordonnées d'onde sont des coordonnées x^μ telles que

$$H^\mu \equiv \mathbf{D}_\alpha \mathbf{D}^\alpha x^\mu = 0. \tag{1.3.3}$$

H^μ peut s'exprimer en fonction de $\mathbf{g}$:

$$H^\mu = \partial_\alpha \mathbf{g}^{\alpha\mu} + \frac{1}{2}\mathbf{g}^{\alpha\mu}\mathbf{g}^{\rho\sigma}\partial_\alpha \mathbf{g}_{\rho\sigma}.$$

Dans les coordonnées d'onde, (1.2.1) s'écrit

$$R^H_{\mu\nu} \equiv -\frac{1}{2}\mathbf{g}^{\alpha\beta}\partial_\alpha\partial_\beta \mathbf{g}_{\mu\nu} + N_{\mu\nu}(\mathbf{g}, \partial\mathbf{g}) = 0, \tag{1.3.4}$$

où les $N_{\mu\nu}$ sont des formes quadratiques en $\partial\mathbf{g}$. (3.6.2) est une équation d'onde quasilinéaire que l'on peut résoudre si on connaît les données initiales $\mathbf{g}_{\mu\nu}(0)$ et $\partial_t \mathbf{g}_{\mu\nu}(0)$ sur Σ. On les construit ainsi :

1. On choisit des coordonnées x^i sur Σ.
2. On prend $\mathbf{g}_{ij}(0) = g_{ij}(0)$ et $\partial_t \mathbf{g}_{ij}(0) = K_{ij}$.
3. On choisit arbitrairement $\mathbf{g}_{00} = -1$ et $\mathbf{g}_{0i} = 0$ (cela correspond à choisir un système de coordonnées d'onde particulier).
4. On prend $\partial_t \mathbf{g}_{0i}$ tel que $H^\mu = 0$ sur Σ.

Une solution de (1.3.4) est une solution de (1.2.1) si et seulement si elle vérifie (1.3.3). Or si les équations de contraintes et (1.3.3) sont vérifiées sur Σ, on a $\partial_t H^\mu = 0$ sur Σ. Les identités de Bianchi et (1.3.4) donnent une équation linéaire hyperbolique d'ordre 2 sur H, qui admet une unique solution. (1.3.3) est donc vérifiée partout. On pourra consulter l'appendice 3.12.2 pour des détails sur les coordonnées d'onde généralisées. Ce choix de coordonnées est celui utilisé par Choquet-Bruhat dans [19] pour montrer l'existence locale pour les équations d'Einstein.

1.3.2 La stabilité de l'espace-temps de Minkowski

Le théorème 1.2.1 ne dit rien sur le comportement en temps long des solutions des équations d'Einstein. Les cas qui pourraient se présenter sont les suivants : formation de singularités, formation d'un horizon de Cauchy (horizon au delà duquel la solution peut être prolongée, mais de manière non déterministe), ou au contraire existence globale : on dit alors que l'espace-temps est géodésiquement complet. Les équations d'Einstein étant d'une grande complexité, un premier problème est d'étudier la stabilité de solutions particulières bien connues, comme par exemple la solution de Minkowski.

La stabilité de Minkowski a été résolue par l'affirmative par Christodoulou et Klainerman dans les années 1990 dans [13]. Leur résultat est le suivant : étant données des données initiales (Σ, g, K)

asymptotiquement plates, suffisamment régulières et satisfaisant une condition de petitesse, il existe un développement globalement hyperbolique, solution des équations d'Einstein dans le vide, géodésiquement complet et feuilleté de manière régulière par des hypersurfaces maximales ($\tau = 0$) et de shift nul ($\beta = 0$).

La stabilité de Minkowski a ensuite été redémontrée par Lindblad et Rodnianski dans [40] en utilisant les coordonnées d'onde. Leur résultat s'étend aux équations d'Einstein couplées à un champ scalaire. Une intuition des difficultés rencontrées sera donnée dans la partie suivante.

1.4 Problèmes d'existence globale à petite donnée initiale pour les équations d'onde non linéaires

Prouver l'existence de solutions globales régulières pour des équations d'ondes nonlinéaires avec de petites données initiales constitue un domaine de recherche important de ces trente dernières années. Cette étude s'appuie sur deux principes très généraux :

- Les solutions des équations d'onde ont tendance à décroître en fonction du temps, et ce d'autant plus que la dimension est élevée (très heuristiquement, plus il y a de dimensions, plus il y a de directions où décroitre). Par exemple, une solution u de l'équation des ondes $\Box u = 0$, pour des données initiales suffisamment régulières a le taux de décroissance
$$u \sim \frac{1}{t^{\frac{n-1}{2}}} \text{ quand } t \to \infty.$$
- On peut écrire le problème comme une perturbation non linéaire d'une équation linéaire : si la donnée initiale est petite, la perturbation est d'autant plus sous contrôle que l'ordre de la non linéarité est élevé.

Dans cette section, on va rappeler quelques méthodes et résultats dans le cas modèle d'équations d'ondes semilinéaires du type $\Box u = (\partial u)^p$.

1.4.1 Estimation d'énergie

On considère le système sur $\mathbb{R}^{n+1}$
$$\begin{cases} \Box u = f, \\ (u, \partial_t u)|_{t=0} = (u_0, u_1). \end{cases} \tag{1.4.1}$$
On peut remarquer que l'opérateur d'Alembertien $\Box$ correspond à $D^\alpha D_\alpha$ où D est la connexion de Levi-Civita associée à la métrique de Minkowski m. On introduit le tenseur énergie impulsion de u
$$Q_{\alpha\beta} = \partial_\alpha u \partial_\beta u - \frac{1}{2} m_{\alpha\beta} m^{\mu\nu} \partial_\mu u \partial_\nu u.$$
On peut calculer
$$D^\alpha Q_{\alpha\beta} = f \partial_\beta u.$$
On note $T = \partial_t$, et on introduit le tenseur de déformation de T
$$^{(T)}\pi_{\alpha\beta} = D_\alpha T_\beta + D_\beta T_\alpha.$$
Comme T est un champ de Killing pour l'espace-temps de Minkowski (voir l'exemple 1.1.13), on a $^{(T)}\pi_{\alpha\beta} = 0$ et par conséquent
$$D^\alpha (Q_{\alpha\beta} T^\beta) = f \partial_t u. \tag{1.4.2}$$

En intégrant l'équation (1.4.2) sur $[0,t]\times\mathbb{R}^n$ on obtient

$$\int_{\mathbb{R}^n} Q_{TT}(t,x)dx = \int_{\mathbb{R}^n} Q_{TT}(0,x)dx + \int_0^t \int_{\mathbb{R}^n} (f\partial_t u)(s,x)dsdx, \tag{1.4.3}$$

où

$$Q_{TT} = Q_{\alpha\beta}T^\alpha T^\beta = \frac{1}{2}\left((\partial_t u)^2 + |\nabla u|^2\right).$$

Remarque 1.4.1. *En appliquant le même procédé aux autres champs de Killing de l'espace temps de Minkowski (voir l'exemple 1.1.13) on obtient d'autre lois de conservations. C'est une application du théorème de Noether qui associe à une symétrie une loi de conservation On suppose pour simplifier $f = 0$. On obtient par exemple*

- *La loi de conservation correspondant aux translations ∂_i est*

$$\int_{\mathbb{R}^n} (\partial_t u \partial_i u)(t,x)dx = \int_{\mathbb{R}^n} (\partial_t u \partial_i u)(0,x)dx,$$

- *La loi de conservation correspondant aux rotations Ω_{ij} est*

$$\int_{\mathbb{R}^n} (\partial_t u \Omega_{ij} u)(t,x)dx = \int_{\mathbb{R}^n} (\partial_t u \Omega_{ij} u)(0,x)dx.$$

1.4.2 La méthode des champs de vecteurs de Klainerman

Les champs de Killing et Killing conformes de l'espace-temps de Minkowski permettent aussi d'obtenir des informations importantes sur le taux de décroissance des solutions. On note

$$\mathcal{Z} = \{\partial_\alpha, \Omega_{\alpha\beta} = -x_\alpha\partial_\beta + x_\beta\partial_\alpha, S = t\partial_t + r\partial_r\},$$

où $x_\alpha = m_{\alpha\beta}x^\beta$. Ces champs de vecteurs satisfont la propriété de commutation suivante

$$[\Box, Z] = C(Z)\Box,$$

où

$$C(Z) = 0,\ Z \neq S, \quad C(S) = 2.$$

Ainsi, si $\Box u = 0$, $\Box Z u = 0$ et l'estimation d'énergie (1.4.3) donne

$$\int \left(\left(\frac{d}{dt}Z^I u\right)^2 + |\nabla Z^I u|^2\right)(t,x)dx = \int \left(\left(\frac{d}{dt}Z^I u\right)^2 + |\nabla Z^I u|^2\right)(0,x)dx \tag{1.4.4}$$

où $Z^I u$ dénote n'importe quelle combinaison de I champs de vecteurs de $\mathcal{Z}$.

L'estimation suivante, appelée estimation de Klainerman-Sobolev, permet d'obtenir une information plus précise que l'injection de Sobolev $H^s \subset L^\infty$ pour $s > \frac{n}{2}$ lorsque l'on contrôle les normes L^2 de $Z^I u$ (ce qui résulte de (1.4.4)). Elle s'écrit

$$(1+t+|x|)^{\frac{n-1}{2}}(1+|t-|x||)^{\frac{1}{2}}|v(t,x)| \leq C \sum_{|I|\leq \frac{n}{2}+1} \|Z^I v\|_{L^2}. \tag{1.4.5}$$

On retrouve ainsi le taux de décroissance $u \sim t^{-\frac{n-1}{2}}$ pour une solution de $\Box u = 0$. Par ailleurs, un calcul simple nous donne.

$$\partial_t + \partial_r = \frac{S + \sum_{i=1}^n \frac{x_i}{r}\Omega_{0i}}{t+r},$$
$$\partial_t - \partial_r = \frac{S - \sum_{i=1}^n \frac{x_i}{r}\Omega_{0i}}{t-r}.$$

Ainsi, les dérivées tangentes au cône de lumière, que l'on note $\bar{\partial}$, ont un meilleur taux de décroissance en temps, donné par $\bar{\partial} u \sim t^{-\frac{n+1}{2}}$.

1.4.3 Application au cas non-linéaire

Nous allons terminer ce paragraphe en essayant de donner une intuition des résultats qu'il est possible d'obtenir dans le cas non linéaire. On considère le problème non linéaire

$$\begin{cases} \Box u = (\partial u)^p, \\ (u, \partial_t u)|_{t=0} = (u_0, u_1), \end{cases} \tag{1.4.6}$$

avec les données initiales (u_0, u_1) de taille ε petites L'estimation d'énergie (1.4.3) nous donne

$$\int_{\mathbb{R}^n} (\partial u)^2(t,x)dx \leq \int_{\mathbb{R}^n} (\partial u)^2(t,0)dx + \int_0^t \int (\partial u)^{p+1}(x,s)dxds.$$

En supposant les estimations a priori suivantes, compatibles avec le comportement de l'équation des ondes linéaires :

$$\int_{\mathbb{R}^n} (\partial u)^2(t,x)dx \leq C\varepsilon, \quad |\partial u| \leq \frac{C\varepsilon}{(1+t)^{\frac{n-1}{2}}},$$

on obtient

$$\int_{\mathbb{R}^n} (\partial u)^2(t,x)dx \leq \int_{\mathbb{R}^n} (\partial u)^2(t,0)dx + C^{p+1}\varepsilon^{p+1} \int_0^t \int \frac{1}{(1+s)^{(p-1)\frac{n-1}{2}}} dxds,$$

ainsi, si $(p-1)\frac{n-1}{2} > 1$, l'intégrale d'espace-temps au second membre converge, ce qui est le point de départ pour permettre de montrer l'existence globale (voir par exemple [33] pour le cas $n > 3$ et $p = 2$). Dans le cas contraire, on peut trouver des contre exemples à l'existence globale (voir [31]). Cependant, dans le cas où la non linéarité présente de la structure, il est possible d'obtenir un meilleur résultat. Supposons par exemple $n = 3$ et $p = 2$. On a alors $(p-1)\frac{n-1}{2} = 1$, et donc pas d'existence globale à priori. Cependant, si la non linéarité peut se factoriser sous la forme $\partial u \bar{\partial} u$, l'estimation d'énergie ainsi que l'estimation à priori pour les dérivées tangentielles

$$|\bar{\partial} u| \leq \frac{C\varepsilon}{(1+t)^2},$$

nous donne

$$\int_{\mathbb{R}^3} (\partial u)^2(t,x)dx \leq \int_{\mathbb{R}^3} (\partial u)^2(t,0)dx + C^2\varepsilon^2 \int_0^t \int \frac{1}{(1+s)^2} dxds,$$

ce qui suggère qu'il y a existence globale dans ce cas. C'est le cas pour les systèmes de la forme

$$\Box u^i = P^i(\partial u^j, \partial u^k), \tag{1.4.7}$$

où les P^i satisfont la condition nulle, introduite par Klainerman dans [32]. Cette condition consiste à choisir des P^i combinaisons linéaires des formes suivantes

$$Q_0(\partial u, \partial v) = \partial_t u \partial_t v - \nabla u.\nabla v, \quad Q_{\alpha\beta}(\partial u, \partial v) = \partial_\alpha u \partial_\beta v - \partial_\alpha v \partial_\beta u.$$

Dans ce cas, on peut montrer qu' il y a existence globale de solutions à (1.4.7) pour des petites données initiales en dimension $3+1$ (voir [32]).

Cependant, la condition nulle n'est pas une condition nécessaire pour avoir l'existence globale de solutions. Un exemple est donnée par le système suivant.

$$\begin{cases} \Box \varphi_1 = 0, \\ \Box \varphi_2 = (\partial_t \varphi_1)^2. \end{cases} \tag{1.4.8}$$

Le découplage permet de résoudre (1.4.8). Cependant, φ_2 a le taux de décroissance $\varphi_2 \sim t^{-1}\ln(t)$, qui est moins que le comportement en t^{-1} de l'équation linéaire correspondante. Il en est de même pour l'équation

$$\partial_t^2 u - u\Delta u,$$

étudiée dans [2] et [38]. Ces deux exemples possèdent une structure appelée structure nulle faible et introduite dans [39]. Lindblad et Rodnianski ont montré la stabilité de Minkowski en coordonnées d'onde, en montrant que les équations d'Einstein dans ces coordonnées ont la structure nulle faible (voir [40]). Ce type de structure nous permettra de montrer la stabilité de l'espace-temps de Minkowski avec un champ de Killing de translation en temps exponentiellement grand dans le Chapitre 3.

1.5 Les équations d'Einstein avec un champ de Killing de translation spatiale

Dans cette section nous présentons les équations d'Einstein avec un champ de Killing de translation spatiale, modèle qui fait l'objet de cette thèse. Les variétés qui nous intéressent sont de la forme $M = \Sigma \times \mathbb{R}_{x_3} \times \mathbb{R}_t$, munies d'une métrique $\mathbf{g}$. On suppose que le champ de vecteur ∂_{x_3} est un champ de Killing. La composante $\mathbb{R}_{x_3}$ peut si on le souhaite, être compactifiée en $\mathbb{S}^1$. Cette symétrie a été introduite par Choquet-Bruhat et Moncrief. Dans [11], ils montrent la stabilité d'une solution particulière, quand Σ est une variété compacte de genre plus grand que 2, correspondant à un univers en expansion. Dans cette thèse, nous considérons dans les chapitres 2 et 3 le cas où Σ est asymptotiquement plate, et dans le chapitre 4 le cas où Σ est une variété compacte de genre plus grand que 2.

1.5.1 Réduction des équations d'Einstein avec un champ de Killing de translation

Dans cette section, nous allons montrer comment les équations d'Einstein dans le vide se transforment en présence d'un champ de Killing de translation. Nous suivons la dérivation faite dans [8], chapitres 6 et 16 et appendice 7.

1.5.1.1 Le tenseur de Ricci dans un repère mobile

On se place sur une variété M de dimension n. Un repère mobile est donné par des champs de vecteurs linéairement indépendants e_β. Un corepère est donné par des 1-formes différentielles θ^α. On dit que le corepère θ^α est le corepère dual de e_β si $\theta^\alpha(e_\beta) = \delta^\alpha_\beta$.

Par exemple, pour le repère donné par les vecteurs coordonnées $\frac{\partial}{\partial x^\alpha}$, le corepère dual est donné par les formes différentielles dx^α. Dans ce cas on a $d\theta^\alpha \equiv 0$.

Dans le cas général on pose

Définition 1.5.1.
$$d\theta^\alpha = -\frac{1}{2} C^\alpha_{\beta\gamma} \theta^\beta \wedge \theta^\gamma.$$
$C^\alpha_{\beta\gamma}$ est appelé coefficient de structure.

Si e_β est la base duale, on a $[e_\alpha, e_\beta] = C^\gamma_{\alpha\beta} e_\gamma$.

On écrit
$$df \equiv \partial_\alpha f \theta^\alpha.$$

Une connexion est la donnée d'une 1-forme à valeur dans les matrices, $\omega^\beta_\gamma = \omega^\beta_{\alpha\gamma}\theta^\alpha$. La dérivée covariante associée est alors donnée par
$$D_\alpha v^\beta = \partial_\alpha v^\beta + \omega^\beta_{\alpha\gamma} v^\gamma.$$

On munit maintenant M d'une métrique g, on peut alors définir la connexion riemannienne. Elle doit avoir les propriétés suivantes

- être sans torsion : $\omega^\alpha_{\beta\gamma} - \omega^\alpha_{\gamma\beta} = C^\alpha_{\beta\gamma}$,

- laisser g invariante : $\partial_\alpha g_{\beta\gamma} - \omega^\lambda_{\alpha\gamma} g_{\lambda\beta} - \omega^\lambda_{\alpha\beta} g_{\lambda\gamma} = 0$.

La connexion est alors donnée par

$$\omega^\beta_{\alpha\gamma} = \Gamma^\beta_{\alpha\gamma} + g^{\beta\mu}\tilde{\omega}_{\alpha\gamma\mu},$$

où on a noté

$$\Gamma^\beta_{\alpha\gamma} = \frac{1}{2} g^{\beta\mu}(\partial_\alpha g_{\mu\gamma} + \partial_\gamma g_{\mu\alpha} - \partial_\mu g_{\alpha\gamma}),$$

$$\tilde{\omega}_{\alpha\gamma\mu} = \frac{1}{2}(g_{\mu\lambda} C^\lambda_{\alpha\gamma} - g_{\lambda\gamma} C^\lambda_{\alpha\mu} - g_{\alpha\lambda} C^\lambda_{\gamma\mu}).$$

On notera aussi

$$\tilde{\omega}^\beta_{\alpha\gamma} = g^{\beta\mu}\tilde{\omega}_{\alpha\gamma\mu}.$$

Proposition 1.5.2. *Dans un repère mobile, le tenseur de Ricci est donné par la formule*

$$R_{\alpha\beta} = \partial_\lambda \omega^\lambda_{\alpha\beta} - \partial_\alpha \omega^\lambda_{\lambda\beta} + \omega^\lambda_{\alpha\beta}\omega^\rho_{\rho\lambda} - \omega^\lambda_{\alpha\rho}\omega^\rho_{\lambda\beta} - \omega^\lambda_{\rho\beta} C^\rho_{\lambda\alpha}.$$

1.5.1.2 Réduction à la Kaluza Klein

On s'intéresse à une variété lorentzienne $(\widehat{V}, \widehat{g})$ de dimension 4 sur laquelle $\mathbb{S}^1$ agit par isométries. On écrit $\widehat{V}$sous la forme sous la forme $V \times \mathbb{S}^1$. On va considérer sur $\widehat{V}$ le repère mobile $(\widehat{e}_\alpha, \widehat{e}_3)$, où $\widehat{e}_3 = \frac{\partial}{\partial x^3}$, avec x^3 la coordonnée selon $\mathbb{S}^1$. $\widehat{e}_3$ est alors un champ de Killing. On choisit ensuite les $\widehat{e}_\alpha$ orthogonaux à $\widehat{e}_3$. Les $\widehat{e}_\alpha$ s'écrivent donc

$$\widehat{e}_\alpha = e_\alpha - A_\alpha \widehat{e}_3,$$

où e_α est un repère sur V. On considère $\widehat{\theta}^\alpha$ le corepère dual. On a alors $\widehat{\theta}^\alpha = \theta^\alpha$ et

$$\widehat{\theta}^3 = \theta^3 + A_\alpha \theta^\alpha,$$

où θ^α est la base duale de e_α et $\theta^3 = dx^3$. On peut donc écrire la métrique $\widehat{g}$ sous la forme

$$\widehat{g} = \widehat{g}_{AB}\widehat{\theta}^A\widehat{\theta}^B = g_{\alpha\beta}\theta^\alpha\theta^\beta + e^{2\varphi}(\theta^3 + A_\alpha\theta^\alpha)^2,$$

où on note A, B, C les indices sur $\widehat{V}$, et α, β, γ les indices sur V.

On note F la 2-forme linéaire alternée définie par

$$F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha.$$

Proposition 1.5.3. *Les coefficients de Ricci s'écrivent, dans la base* $(\widehat{\theta}^\alpha, \widehat{\theta}^3)$

$$\widehat{R}_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2} e^{2\varphi} F_\alpha{}^\lambda F_{\beta\lambda} - D_\alpha \partial_\beta \varphi - \partial_\alpha \varphi \partial_\beta \varphi, \tag{1.5.1}$$

$$\widehat{R}_{\alpha 3} = \frac{1}{2} e^{-\varphi} D_\beta (e^{3\varphi} F_\alpha{}^\beta), \tag{1.5.2}$$

$$\widehat{R}_{33} = -e^{-2\varphi}\left(-\frac{1}{4} e^{2\varphi} F_{\alpha\beta}F^{\alpha\beta} + g^{\alpha\beta}\partial_\alpha\varphi\partial_\beta\varphi + g^{\alpha\beta} D_\alpha \partial_\beta \varphi\right). \tag{1.5.3}$$

Démonstration. On va commencer par calculer les coefficients de structure. On suppose pour simplifier que les e_α sont des vecteurs coordonnées de V. On a alors $\widehat{\theta}^\alpha = dx^\alpha$, et donc $d\widehat{\theta}^\alpha = 0$ et $\widehat{C}^\alpha_{BC} = 0$. Par ailleurs

$$d\widehat{\theta}^3 = d(A_\alpha \theta^\alpha) = dA_\alpha \wedge \theta^\alpha = \frac{1}{2}(\partial_\alpha A_\beta - \partial_\beta A_\alpha) dx^\alpha \wedge dx^\beta.$$

On a donc $\widehat{C}^3_{\alpha\beta} = -F_{\alpha\beta}$ et $\widehat{C}^3_{3A} = 0$.

On calcule ensuite les coefficients de connexion $\widehat{\omega}_{BC}^{A} = \widehat{\Gamma}_{BC}^{A} + \widetilde{\omega}_{BC}^{A}$. On calcule aisément les coefficients $\widehat{\Gamma}$ en remarquant que $\partial_3 g_{AB} = \partial_3 \varphi = \partial_3 A_\alpha = 0$. On a

$$\widehat{\Gamma}_{33}^{3} = \widehat{\Gamma}_{\alpha\beta}^{3} = \widehat{\Gamma}_{3\beta}^{\alpha} = 0,$$

$$\widehat{\Gamma}_{\beta\varphi}^{\alpha} = \Gamma_{\beta\varphi}^{\alpha}, \quad \widehat{\Gamma}_{\alpha 3}^{3} = \frac{1}{2} e^{-2\varphi} \partial_\alpha e^{2\varphi}, \quad \widehat{\Gamma}_{33}^{\alpha} = -\frac{1}{2} \partial^\alpha e^{2\varphi}.$$

On a par ailleurs, au vu des coefficients de structure

$$\widetilde{\omega}_{\beta\varphi}^{\alpha} = \widetilde{\omega}_{33}^{A} = \widetilde{\omega}_{A3}^{3} = \widetilde{\omega}_{3A}^{3} = 0,$$

$$\widetilde{\omega}_{\alpha\beta}^{3} = -\frac{1}{2} F_{\alpha\beta}, \quad \widetilde{\omega}_{3\beta}^{\alpha} = \widetilde{\omega}_{\beta 3}^{\alpha} = -\frac{1}{2} e^{2\varphi} F^{\alpha}{}_{\beta}.$$

On peut donc calculer les coefficients de Ricci

$$\begin{aligned}
\widehat{R}_{\alpha\beta} = & R_{\alpha\beta} - \partial_\alpha \widehat{\omega}_{3\beta}^{3} + \widehat{\omega}_{\alpha\beta}^{3} \widehat{\omega}_{\rho 3}^{\rho} + \widehat{\omega}_{\alpha\beta}^{\rho} \widehat{\omega}_{3\rho}^{3} - \widehat{\omega}_{\alpha\rho}^{3} \widehat{\omega}_{3\beta}^{\rho} - \widehat{\omega}_{\alpha 3}^{\lambda} \widehat{\omega}_{\lambda\beta}^{3} - \widehat{\omega}_{\alpha 3}^{3} \widehat{\omega}_{3\beta}^{3} - \widehat{\omega}_{3\beta}^{\lambda} \widehat{C}_{\lambda\alpha}^{3} \\
= & R_{\alpha\beta} - \frac{1}{2} \partial_\alpha (e^{-2\varphi} \partial_\beta e^{2\varphi}) + \frac{1}{2} \Gamma_{\alpha\beta}^{\rho} e^{-2\varphi} \partial_\rho e^{2\varphi} - \frac{1}{4} e^{2\varphi} F_{\alpha\rho} F^{\rho}{}_{\beta} - \frac{1}{4} e^{2\varphi} F^{\lambda}{}_{\alpha} F_{\lambda\beta} \\
& - \frac{1}{4} e^{-4\varphi} \partial_\alpha e^{2\varphi} \partial_\beta e^{2\varphi} - \frac{1}{2} e^{2\varphi} F^{\lambda}{}_{\beta} F_{\lambda\alpha} \\
= & R_{\alpha\beta} - D_\alpha \partial_\beta \varphi - \partial_\alpha \varphi \partial_\beta \varphi - \frac{1}{2} e^{2\varphi} F^{\lambda}{}_{\alpha} F_{\lambda\beta},
\end{aligned}$$

$$\begin{aligned}
\widehat{R}_{3\alpha} = & \partial_\lambda \widehat{\omega}_{3\alpha}^{\lambda} + \widehat{\omega}_{3\alpha}^{B} \widehat{\omega}_{CB}^{C} - \widehat{\omega}_{3C}^{B} \widehat{\omega}_{B\alpha}^{C} - \widehat{\omega}_{33}^{\rho} \widehat{C}_{\rho\alpha}^{3} \\
= & -\frac{1}{2} \partial_\lambda (e^{2\varphi} F^{\lambda}{}_{\alpha}) - \frac{1}{2} e^{2\varphi} F^{\beta}{}_{\alpha} \left(\Gamma_{\rho\beta}^{\rho} + \frac{1}{2} e^{-2\varphi} \partial_\beta e^{2\varphi} \right) + \frac{1}{2} e^{2\varphi} F^{\beta}{}_{\varphi} \Gamma_{\beta\alpha}^{\varphi} + \frac{1}{4} \partial_\rho e^{2\varphi} F^{\rho}{}_{\alpha} \\
& - \frac{1}{4} \partial^\beta e^{2\varphi} F_{\alpha\beta} - \frac{1}{2} \partial^\rho e^{2\varphi} F_{\rho\alpha} \\
= & -\frac{1}{2} e^{2\varphi} D_\lambda F^{\lambda}{}_{\alpha} - \frac{3}{4} F^{\lambda}{}_{\alpha} \partial_\lambda e^{2\varphi} \\
= & -\frac{1}{2} e^{-\varphi} D_\lambda (e^{3\varphi} F^{\lambda}{}_{\alpha}),
\end{aligned}$$

$$\begin{aligned}
\widehat{R}_{33} = & \partial_\lambda \widehat{\omega}_{33}^{\lambda} + \widehat{\omega}_{33}^{C} \widehat{\omega}_{BC}^{B} - \widehat{\omega}_{3C}^{B} \widehat{\omega}_{B3}^{C} \\
= & -\frac{1}{2} \partial_\lambda \partial^\lambda e^{2\varphi} - \frac{1}{2} \partial^\rho e^{2\varphi} \left(\Gamma_{\beta\rho}^{\beta} + \frac{1}{2} e^{-2\varphi} \partial_\rho e^{2\varphi} \right) + \frac{1}{2} e^{-2\varphi} \partial^\beta e^{2\varphi} \partial_\beta e^{2\varphi} - \frac{1}{4} e^{4\varphi} F^{\alpha}{}_{\beta} F^{\beta}{}_{\alpha} \\
= & -e^{2\varphi} \left(D^\lambda \partial_\lambda \varphi + \partial^\lambda \varphi \partial_\lambda \varphi - \frac{1}{4} e^{2\varphi} F_{\alpha\beta} F^{\alpha\beta} \right).
\end{aligned}$$

Ceci conclut la preuve de la proposition. □

1.5.1.3 Potentiel twist

Définition 1.5.4. *L'étoile de Hodge, notée $*$ est l'application qui à une k-forme linéaire alternée α, associe la $n-k$ forme alternée $*\alpha$ telle que pour toute k-forme linéaire alternée β on ait*

$$\beta \wedge *\alpha = <\beta, \alpha> dvol_g,$$

où on a noté $dvol_g$ la forme volume et $<,>$ le produit scalaire défini par g.

Sur une variété lorentzienne de dimension 3, on obtient alors, en prenant $\theta^0, \theta^1, \theta^2$ une base orthonormale de formes linéaires telles que $g(\theta^0, \theta^0) = -1$:

$$*\theta^0 = -\theta^1 \wedge \theta^2, \quad *\theta^1 = -\theta^0 \wedge \theta^2, \quad *\theta^2 = \theta^0 \wedge \theta^1.$$

On suppose que $\widehat{g}$ satisfait les équations d'Einstein $\widehat{R}_{AB} = 0$. L'équation $\widehat{R}_{3\alpha} = 0$ peut s'écrire, d'après (1.5.2)

$$d(e^{3\varphi} * F) = 0.$$

Par le théorème de dualité de Poincaré, il existe donc une fonction ω telle que $e^{3\varphi} * F = d\omega$, ce qui s'écrit encore

$$F = -e^{-3\varphi} * d\omega.$$

Comme $F = dA$, on a $dF = 0$ donc

$$D^\alpha \partial_\alpha \omega - 3\partial_\alpha \omega \partial^\alpha \varphi = 0. \tag{1.5.4}$$

Pour les autres équations, on va avoir besoin du lemme suivant.

Lemme 1.5.5. *On a l'égalité suivante*

$$F_{\alpha\lambda} {F_\beta}^\lambda = e^{-6\varphi} \frac{1}{4} (\partial_\alpha \omega \partial_\beta \omega - g_{\alpha\beta} \partial_\lambda \omega \partial^\lambda \omega).$$

Démonstration. On calcule les termes un par un dans la base $\theta^0, \theta^1, \theta^2$.

$$\begin{aligned}
{F_0}^\lambda F_{1_\lambda} &= F_{02} F_{12} = \frac{1}{4} e^{-6\varphi} \partial_0 \omega \partial_1 \omega, \\
{F_0}^\lambda F_{2_\lambda} &= F_{01} F_{21} = \frac{1}{4} e^{-6\varphi} \partial_0 \omega \partial_2 \omega, \\
{F_1}^\lambda F_{2_\lambda} &= -F_{10} F_{20} = \frac{1}{4} e^{-6\varphi} \partial_1 \omega \partial_2 \omega, \\
{F_0}^\lambda F_{0\lambda} &= F_{01} F_{01} + F_{02} F_{02} = \frac{1}{4} e^{-6\varphi} ((\partial_1 \omega)^2 + (\partial_2 \omega^2)) = \frac{1}{4} e^{-6\varphi} (\partial_\lambda \omega \partial^\lambda \omega + (\partial_0 \omega)^2), \\
{F_1}^\lambda F_{1\lambda} &= -F_{10} F_{10} + F_{12} F_{12} = \frac{1}{4} e^{-6\varphi} ((\partial_0 \omega)^2 - (\partial_2 \omega^2)) = \frac{1}{4} e^{-6\varphi} (-\partial_\lambda \omega \partial^\lambda \omega + (\partial_1 \omega)^2), \\
{F_2}^\lambda F_{2\lambda} &= -F_{20} F_{20} + F_{21} F_{21} = \frac{1}{4} e^{-6\varphi} ((\partial_0 \omega)^2 - (\partial_1 \omega^2)) = \frac{1}{4} e^{-6\varphi} (-\partial_\lambda \omega \partial^\lambda \omega + (\partial_2 \omega)^2).
\end{aligned}$$

Ceci conclut la preuve du lemme. □

Les équations $\widehat{R}_{\alpha\beta} = 0$ et $\widehat{R}_{33} = 0$ s'écrivent alors

$$R_{\alpha\beta} - \frac{1}{8} e^{-4\varphi} (\partial_\alpha \omega \partial_\beta \omega - g_{\alpha\beta} \partial_\lambda \omega \partial^\lambda \omega) - D_\alpha \partial_\beta \varphi - \partial_\alpha \varphi \partial_\beta \varphi = 0, \tag{1.5.5}$$

$$\frac{1}{8} e^{-4\varphi} \partial_\lambda \omega \partial^\lambda \omega + g^{\alpha\beta} \partial_\alpha \varphi \partial_\beta \varphi + g^{\alpha\beta} D_\alpha \partial_\beta \varphi = 0. \tag{1.5.6}$$

1.5.1.4 Transformation conforme

Pour mettre les équations sous une forme plus agréable, on va faire la transformation conforme $\tilde{g} = e^{2\varphi} g$. On a alors, grâce à (1.2.4)

$$D^\rho \partial_\rho \omega = e^{2\varphi} (\widetilde{D}^\rho \partial_\rho \omega + e^{2\varphi} \widetilde{g}^{\rho\beta} (\delta^\alpha_\beta \partial_\rho \varphi + \delta^\alpha_\rho \partial_\beta \varphi - \widetilde{g}_{\beta\rho} \partial^\alpha \varphi) \partial_\alpha \omega = e^{2\varphi} \widetilde{D}^\rho \partial_\rho \omega - e^{2\varphi} \partial_\rho \varphi \partial^\rho \omega,$$

où les indices sont montés avec $\widetilde{g}$. L'équation (1.5.4) devient alors

$$\widetilde{D}^\alpha \partial_\alpha \omega - 4\partial_\alpha \omega \partial^\alpha \varphi = 0. \tag{1.5.7}$$

L'équation (1.5.6) devient

$$\frac{1}{8}e^{-6\varphi}\partial_\lambda\omega\partial^\lambda\omega + e^{-2\varphi}\widetilde{g}^{\alpha\beta}\partial_\alpha\varphi\partial_\beta\varphi + e^{-2\varphi}\widetilde{g}^{\alpha\beta}(\widetilde{D}_\alpha\partial_\beta\varphi + 2\partial_\alpha\varphi\partial_\beta\varphi - \widetilde{g}_{\alpha\beta}\partial_\rho\varphi\partial^\rho\varphi = 0,$$

ce qui s'écrit

$$\widetilde{D}^\rho\partial_\rho\varphi + \frac{1}{8}e^{-4\varphi}\partial_\lambda\omega\partial^\lambda\omega = 0. \tag{1.5.8}$$

Grâce à la proposition 1.2.1 l'équation (1.5.5) devient

$$\begin{aligned} 0 =& \widetilde{R}_{\alpha\beta} + \widetilde{g}_{\alpha\beta}\widetilde{D}^\lambda\partial_\lambda\varphi + \widetilde{D}_\alpha\partial_\beta\varphi + (\partial_\alpha\varphi\partial_\beta\varphi - \widetilde{g}_{\alpha\beta}\widetilde{g}^{\lambda\rho}\partial_\rho\varphi\partial_\lambda\varphi) \\ &- \frac{1}{8}e^{-4\varphi}(\partial_\alpha\omega\partial_\beta\omega - g_{\alpha\beta}\partial_\lambda\omega\partial^\lambda\omega) - D_\alpha\partial_\beta\varphi - (2\partial_\alpha\varphi\partial_\beta\varphi - \widetilde{g}_{\alpha\beta}\partial_\rho\varphi\partial^\rho\varphi) - \partial_\alpha\varphi\partial_\beta\varphi. \end{aligned}$$

En remplaçant $D^\lambda\partial_\lambda\varphi$ par sa valeur obtenue dans (1.5.8) on obtient

$$\widetilde{R}_{\alpha\beta} - \frac{1}{8}e^{-4\varphi}\partial_\alpha\omega\partial_\beta\omega - 2\partial_\alpha\varphi\partial_\beta\varphi = 0. \tag{1.5.9}$$

Les équations $\widehat{R}_{AB} = 0$ sont donc équivalentes, en présence d'un champ de Killing de translation spatiale au système

$$\begin{cases} \Box_g\varphi + \frac{1}{8}e^{-4\varphi}\partial^\rho\omega\partial_\rho\omega = 0, \\ \Box_g\omega - 4\partial^\rho\omega\partial_\rho\varphi = 0, \\ R_{\alpha\beta} = 2\partial_\alpha\varphi\partial_\beta\varphi + \frac{1}{8}e^{-4\varphi}\partial_\alpha\omega\partial_\beta\omega. \end{cases} \tag{1.5.10}$$

C'est le système auquel on s'intéressera dans toute cette thèse.

1.5.2 Les ondes d'Einstein Rosen

Terminons cette section en présentant l'exemple instructif des ondes d'Einstein-Rosen, qui sont des solutions particulières des équations d'Einstein dans le vide avec deux champs de Killing orthogonaux : ∂_3 et ∂_θ. Elles ont été découvertes pour la première fois par Beck dans [5]. La métrique en dimension $3+1$ peut s'écrire

$$\mathbf{g} = e^{2\varphi}(dx^3)^2 + e^{2(a-\varphi)}(-dt^2 + dr^2) + r^2e^{-2\varphi}r^2d\theta^2.$$

Les équations d'Einstein dans le vide réduites (1.5.10) s'écrivent alors

$$\begin{aligned} R_{tt} &= \partial_r^2 a - \partial_t^2 a + \frac{1}{r}\partial_r a = 2(\partial_t\varphi)^2, \\ R_{rr} &= -\partial_r^2 a + \partial_t^2 a + \frac{1}{r}\partial_r a = 2(\partial_r\varphi)^2, \\ R_{tr} &= \frac{1}{r}\partial_t a = 2\partial_t\varphi\partial_r\varphi, \end{aligned} \tag{1.5.11}$$

où $R_{\alpha\beta}$ est le tenseur de Ricci de la métrique

$$g = e^{2a}(-dt^2 + dr^2) + r^2d\theta^2.$$

Comme φ est radiale, l'équation pour φ devient

$$e^{2a}\Box_g\varphi = -\partial_t^2\varphi + \partial_r^2\varphi + \frac{1}{r}\partial_r\varphi = 0.$$

Cette équation est découplée de celle pour la métrique. Par conséquent, on peut résoudre l'équation d'onde plate homogène $\Box\varphi = 0$, avec données initiales $(\varphi, \partial_t\varphi)|_{t=0} = (\varphi_0, \varphi_1)$ puis résoudre les équations d'Einstein, qui deviennent

$$\partial_r a = r\left((\partial_r\varphi)^2 + (\partial_t\varphi)^2\right), \tag{1.5.12}$$

avec la condition au bord $a|_{r=0} = 0$ afin d'obtenir une solution régulière. On a alors

$$a = \int_0^R r\left((\partial_r\varphi)^2 + (\partial_t\varphi)^2\right) dr = E(\varphi) - \int_R^\infty r\left((\partial_r\varphi)^2 + (\partial_t\varphi)^2\right) dr,$$

où

$$E(\varphi) = \int_0^\infty r\left((\partial_r\varphi)^2 + (\partial_t\varphi)^2\right) dr.$$

La quantité $E(\varphi)$, qui correspond à l'énergie de φ, ne dépend pas du temps. En particulier, la métrique

$$e^{2a}dr^2 + r^2d\theta^2$$

possède un angle à l'infini spatial, c'est à dire que les cercles de rayon r ont un périmètre qui croit asymptotiquement comme $e^{-E(\varphi)}2\pi r$ au lieu de $2\pi r$. Ces solutions particulières vont nous donner une intuition du comportement auquel d'attendre quand on étudiera le cas asymptotiquement plat dans les chapitres 2 et 3.

1.6 Énoncé des résultats

1.6.1 Stabilité en temps exponentiel de l'espace-temps de Minkowski avec un champ de Killing spatial de translation

On s'intéresse ici à des variétés de la forme $(\mathbb{R}^2 \times \mathbb{R}_{x_3} \times \mathbb{R}_t, \mathbf{g})$, où ∂_{x_3} est un champ de Killing. La question principale dans cette thèse est la suivante : l'espace-temps de Minkowski étant une solution des équations d'Einstein dans le vide avec un champ de Killing spatial de translation, a-t-on stabilité de cette solution dans ce cadre là ? Cela revient à se demander si pour des données initiales pour (φ, ω) proches de $(0,0)$ et des données initiales pour g proches des données initiales pour l'espace-temps de Minkowski, solutions des équations de contraintes, on a existence globale pour le système (1.5.10).

Avant de s'intéresser au problème de stabilité, il faut se demander quelles sont les données initiales licites pour ce problème, c'est à dire quelles sont les données initiales satisfaisant les équations de contraintes. On veut considérer des perturbations asymptotiquement plate en dimension $2+1$. Cependant, cette définition n'est pas totalement claire. Rappelons par exemple les ondes d'Einstein-Rosen qui possèdent un angle à l'infini spatial. Dans toute la suite χ sera une fonction telle que telle que $\chi(q) = 0$ pour $q \leq 1$ et $\chi(q) = 1$ pour $q \geq 2$.

1.6.1.1 Les équations de contraintes

Nous décrivons dans cette partie le résultat du Chapitre 2. On note

$$u \equiv (\gamma, \omega), \tag{1.6.1}$$

muni du produit scalaire

$$\partial_\alpha u.\partial_\beta u = 2\partial_\alpha\gamma\partial_\beta\gamma + \frac{1}{2}e^{-4\gamma}\partial_\alpha\omega\partial_\beta\omega. \tag{1.6.2}$$

Dans le chapitre 2, on résout les équations de contraintes pour $(\dot{u}, \nabla u)$ dans des espaces de Sobolev à poids définis par la norme

$$\|u\|_{H^m_\delta} = \sum_{|\beta|\leq m} \|(1+|x|^2)^{\frac{\delta+|\beta|}{2}} D^\beta u\|_{L^2},$$

où $\delta \in \mathbb{R}$ et $m \in \mathbb{N}$. La méthode utilisée s'inspire de la méthode conforme. On cherche les solutions (g, K) des équations de contraintes sous la forme

$$g = e^{2\lambda}\delta, \quad K = H + \frac{1}{2}\tau g,$$

où δ est la métrique euclidienne et H un 2-tenseur symétrique et sans trace. La fonction τ est la courbure moyenne. Les équations de contraintes s'écrivent alors

$$\left\{ \begin{array}{l} \partial_i H_{ij} = -\dot{u}.\partial_j u + \frac{1}{2}e^{2\lambda}\partial_j \tau, \\ \Delta\lambda + e^{-2\lambda}\left(\frac{1}{2}\dot{u}^2 + \frac{1}{2}|H|^2\right) - e^{2\lambda}\frac{\tau^2}{4} + \frac{1}{2}|\nabla u|^2 = 0. \end{array} \right. \tag{1.6.3}$$

Remarque 1.6.1.

- *Les solutions de $\Delta u = f$ sur $\mathbb{R}^2$ divergeant logarithmiquement comme $\left(\int f\right)\ln(r)$, on peut s'attendre à avoir $\lambda \sim -\alpha \ln(r)$ quand r tend vers $+\infty$. Ceci correspond en fait à la présence d'un angle à l'infini spatial, comme dans les ondes d'Einstein-Rosen.*
- *On ne peut pas se placer dans le cas CMC en supposant $\tau = 0$, car la deuxième équation de* (1.6.3) *pourrait dans ce cas ne pas avoir de solution.*
- *Pour résoudre* (1.6.3)*, qui correspond à trois équations elliptiques scalaires, il faut satisfaire trois conditions d'orthogonalité. On va donc choisir l'angle et le comportement asymptotique de τ afin de forcer ces trois conditions d'orthogonalité.*

On introduit la notation

$$M_\theta = \begin{pmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{pmatrix}, \quad N_\theta = \begin{pmatrix} -\sin(2\theta) & \cos(2\theta) \\ \cos(2\theta) & \sin(2\theta) \end{pmatrix}.$$

Dans [29] nous avons prouvé le résultat suivant.

Théorème 1.6.1 ([29])**.** *Soit $-1 < \delta < 0$. On suppose $e^{-2\lambda}\dot{u}^2, |\nabla u|^2 \in H^0_{\delta+2}$, $\widetilde{\tau} \in H^1_{\delta+1}$ et $\widetilde{b} \in L^\infty(\mathbb{S}^1)$ tel que*

$$\int_{\mathbb{S}^1} \widetilde{b}(\theta)\cos(\theta)d\theta = \int_{\mathbb{S}^1} \widetilde{b}(\theta)\sin(\theta)d\theta = 0.$$

Si toutes ces quantités sont assez petites, il existe $\alpha, \rho, \eta \in \mathbb{R}^3$ et $\widetilde{\lambda} \in H^2_\delta$ une fonction, $\widetilde{H} \in H^1_{\delta+1}$ un tenseur symétrique et sans trace tels que

$$\begin{aligned} H &= -\frac{\chi(r)}{2r}e^\lambda\left(\widetilde{b}(\theta) + \rho\cos(\theta-\eta)\right)M_\theta + e^\lambda \widetilde{H}, \\ \lambda &= -\alpha\chi(r)\ln(r) + \widetilde{\lambda}, \end{aligned}$$

est solution de (1.6.3) *avec*

$$\tau = e^{-\lambda}\frac{\chi(r)}{r}\left(\widetilde{b}(\theta) + \rho\cos(\theta-\eta)\right) + e^{-\lambda}\widetilde{\tau}.$$

Ce théorème n'est pas prouvé dans cette thèse, car le résultat et les techniques sont très semblables au résultat suivant, qui permet de préciser le développement asymptotique des solutions dans le cas où les données pour u ont plus de décroissance. Notons que pour avoir plus de décroissance sur les solutions, il est nécessaire de forcer un plus grand nombre de conditions d'orthogonalité.

Théorème 1.6.2. *Soit $-1 < \delta < 0$. Soient $e^{-2\lambda}\dot{u}^2, |\nabla u|^2 \in H^0_{\delta+3}$ et $\widetilde{b} \in W^{1,2}(\mathbb{S}^1)$ tels que*

$$\int_{\mathbb{S}^1} \widetilde{b}(\theta)\cos(\theta)d\theta = \int_{\mathbb{S}^1} \widetilde{b}(\theta)\sin(\theta)d\theta = 0.$$

Soit $\Psi \in H^1_{\delta+2}$ tel que $\int \Psi = 2\pi$. Si toutes ces données assez petites, il existe $\alpha, \rho, \eta, A, J, c_1, c_2$ dans $\mathbb{R}$, une fonction scalaire $\widetilde{\lambda} \in H^2_{\delta+1}$ et un tenseur symétrique et sans trace $\widetilde{H} \in \mathcal{H}^1_{\delta+2}$ tels que, si r, θ sont les coordonnées polaires centrées en c_1, c_2, et si on note

$$\begin{aligned} \lambda &= -\alpha\chi(r)\ln(r) + \widetilde{\lambda}, \\ H &= -e^\lambda(\widetilde{b}(\theta) + \rho\cos(\theta-\eta))\frac{\chi(r)}{2r}M_\theta + \frac{\chi(r)}{r^2}JN_\theta M_\theta + e^\lambda\widetilde{H}, \end{aligned}$$

alors λ, H *sont solutions de* (1.6.3) *avec*

$$\tau = e^{-\lambda}(\widetilde{b}(\theta) + \rho\cos(\theta-\eta))\frac{\chi(r)}{r} + Ae^{-\lambda}\Psi.$$

De plus, on a les estimations

$$\begin{aligned}
\alpha &\sim \frac{1}{4\pi}\int \left(\dot{u}^2 + |\nabla u|^2\right), \\
\rho\cos(\eta) &\sim \frac{1}{\pi}\int \dot{u}\partial_1 u, \\
\rho\sin(\eta) &\sim \frac{1}{\pi}\int \dot{u}\partial_2 u, \\
c_1 &\sim -\frac{1}{4\pi}\int x_1\left(\dot{u}^2 + |\nabla u|^2\right), \\
c_2 &\sim -\frac{1}{4\pi}\int x_2\left(\dot{u}^2 + |\nabla u|^2\right), \\
J &\sim -\frac{1}{2\pi}\int \dot{u}\partial_\theta u + \frac{\rho}{\alpha}(c_2\cos(\eta) - c_1\sin(\eta)), \\
A &\sim -\frac{1}{2\pi}\int \dot{u}r\partial_r u + \frac{1}{2\pi}\left(\int \chi'(r)rdr\right)\int b(\theta)d\theta.
\end{aligned}$$

Les quantités α, ρ, η, c, J peuvent être rapprochées des charges globales en dimension 3 (voir par exemple l'article de revue [14]). Ces charges globales sont des quantités conservées par le flot, qui correspondent au fait que sur une variété asymptotiquement plate, les champs de Killing de l'espace-temps de Minkowski sont de bonnes symétries approchées à l'infini. Les quantités à droite, dépendant de u correspondent elles aussi à des quantités conservées pour les solutions de $\Box u = 0$ (voir la remarque 1.4.1), ces lois de conservations correspondant au même champs de Killing dans l'espace-temps de Minkowski.

1.6.1.2 Stabilité en temps exponentiel

Nous décrivons dans cette partie le résultat du Chapitre 3. Nous considérons le système 1.5.10 dans le cas particulier $\omega = 0$ pour simplifier l'analyse, mais le résultat est sans doute vrai dans le cas général. Le système devient alors

$$\begin{cases} \Box_g \varphi = 0, \\ R_{\alpha\beta} = \partial_\alpha\varphi\partial_\beta\varphi. \end{cases} \tag{1.6.4}$$

Nous nous plaçons dans les coordonnées d'onde, en s'inspirant de [40]. De la même manière que dans [40], nous exploitons la condition d'onde pour obtenir plus de décroissance sur le coefficient de métrique g_{LL} où $L = \partial_t + \partial_r$. On note aussi $\underline{L} = \partial_t - \partial_r$. Ainsi les termes quasilinéaires $g_{LL}\partial^2_{\underline{L}}\varphi$ et $g_{LL}\partial^2_{\underline{L}}g$, qui ne font intervenir aucune dérivée tangentielle, deviennent contrôlables. Cependant, le fait de travailler en dimension $2+1$ induit de nouvelles difficultés par rapport à [40]. En particulier :

- Moins d'estimations sont disponibles pour étudier l'équation des ondes en dimension $2+1$.
- Le taux de décroissance de la solution de l'équation des ondes $\Box u = 0$ en dimension $2+1$ est $u \sim \frac{1}{\sqrt{t}}$. Ainsi non seulement les termes quadratiques, mais aussi les termes cubiques peuvent faire obstruction à l'existence globale. Le problème principal concerne le système suivant, qui modélise le comportement du coefficient de métrique $g_{\underline{L}\underline{L}}$

$$\begin{cases} \Box\varphi = 0 \\ \Box g_{\underline{L}\underline{L}} = -8(\partial_{\underline{L}}\varphi)^2. \end{cases} \tag{1.6.5}$$

Le taux de décroissance $\varphi \sim \frac{1}{\sqrt{t}}$ ne permet d'obtenir aucun taux de décroissance sur $g_{\underline{L}\underline{L}}$.

- Le comportement asymptotique des données initiales donné par le Théorème 1.6.2 ne permet pas de travailler dans les coordonnées d'onde. On va donc à la place travailler en coordonnées d'onde généralisées.

On introduit la famille de métriques

$$g_b = -dt^2 + dr^2 + (r + \chi(q)b(\theta)q)^2 d\theta^2 + J(\theta)\chi(q)dqd\theta,$$

où on a noté $q = r - t$. Ces métriques sont Ricci plates pour $q \geq 2$ et sont égales à la métrique de Minkowski pour $q \leq 1$. Le terme principal du tenseur de Ricci est

$$(R_b)_{qq} = \frac{\partial_q^2(q\chi(q))}{r}.$$

L'idée principale du chapitre 3 est de s'inspirer de l'analyse des ondes d'Einstein-Rosen de la section 1.5.2 afin d'approcher $\frac{1}{4}g_{\underline{L}\underline{L}}$ par la solution h_0 de l'équation de transport

$$\partial_q h_0 = -2r(\partial_q \varphi) - 2\partial_q^2(q\chi(q))b(\theta).$$

Afin d'obtenir des solutions qui convergent vers Minkowski à l'infini temporel, condition qui est nécessaire pour mener à bien notre analyse, on va chercher à avoir la propriété suivante

$$b(\theta) \sim -\int_0^\infty (\partial_q \varphi)^2(T, r, \theta) r dr. \tag{1.6.6}$$

Le théorème que l'on obtient est le suivant

Théorème 1.6.3. *Soit* $0 < \varepsilon < 1$. *Soient* $\frac{1}{2} < \delta < 1$ *et* $N \geq 40$. *Soit* $(\varphi_0, \varphi_1) \in H_\delta^{N+1}(\mathbb{R}^2) \times H_{\delta+1}^N(\mathbb{R}^2)$. *On suppose*

$$\|\varphi_0\|_{H_\delta^{N+1}} + \|\varphi_1\|_{H_{\delta+1}^N} \leq \varepsilon.$$

Soit $T \leq \exp\left(\frac{1}{\sqrt{\varepsilon}}\right)$. *Si* ε *est assez petit, il existe* $b(\theta), J(\theta) \in W^{N,2}(\mathbb{S}^1)$ *ainsi qu'un système global de coordonnées* (t, x_1, x_2) *tels que, pour* $t \leq T$, *il existe une solution* (φ, g) *de* (1.6.4) *que l'on peut écrire sous la forme*

$$g = g_b + \frac{g_{\underline{L}\underline{L}}}{4} dq^2 + \tilde{g}$$

avec pour tout $\rho > 0$

$$|g_{\underline{L}\underline{L}}| \leq \frac{C\varepsilon}{(1+|q|)^{\frac{1}{2}-\rho}}, \quad |\tilde{g}| \leq \frac{C\varepsilon}{(1+t+r)^{\frac{1}{2}-\rho}}.$$

De plus la condition (1.6.6) *est vérifiée, avec une erreur de taille* $\frac{\varepsilon^2}{\sqrt{T}}$.

Un énoncé précis de ce théorème est donné dans le chapitre 3.

1.6.2 Solution des équations de contraintes dans le cas compact sans la condition CMC

Le chapitre 4 est indépendant des chapitres 2 et 3 et présente un résultat obtenu en collaboration avec Romain Gicquaud. On se place également en présence d'un champ de vecteur de translation spatiale. On s'intéresse à des variétés de la forme $\Sigma \times \mathbb{S}^1 \times \mathbb{R}_t$, munie d'une métrique g, telle ∂_{x_3} est un champ de Killing, où x_3 est la coordonnée selon $\mathbb{S}^1$. Ici Σ sera une variété compacte de genre plus grand que 2. Avec la méthode conforme, le système devient

$$\begin{cases} div(LW) = -\dot{u} \cdot du + \frac{e^{2\lambda}}{2} d\tau, \\ \Delta\lambda + e^{-2\lambda}\left(\frac{1}{2}\dot{u}^2 + \frac{1}{2}|\sigma + LW|^2\right) = e^{2\lambda}\frac{\tau^2}{4} - \frac{1}{2}\left(1 + |\nabla u|^2\right). \end{cases} \tag{1.6.7}$$

Le théorème suivant est l'analogue du résultat de [16], obtenu dansle cas d'une variété compacte en dimension $n \geq 3$.

Théorème 1.6.4. *On suppose* $\dot{u} \in L^\infty(\Sigma, \mathbb{R})$, $u \in W^{1,\infty}(\Sigma, \mathbb{R})$, $\tau \in W^{1,p}(\Sigma, \mathbb{R})$ *et* $\sigma \in W^{1,p}$ *un tenseur symétrique, sans trace et sans divergence, avec* $p > 2$, *et on suppose que* τ *ne s'annule nul part sur* Σ, *alors au moins l'une des assertions suivantes est vraie*

1. *Le système* (1.6.7) *admet au moins une solution* $(\lambda, W) \in W^{2,p}(\Sigma, \mathbb{R}) \times W^{2,p}(\Sigma, T\Sigma)$,
2. *Il existe une solution non triviale* $V \in W^{2,p}(\Sigma, T^*\Sigma)$ *de l'équation limite suivante*

$$div(LW) = \alpha \frac{\sqrt{2}}{2} |LW| \frac{d\tau}{|\tau|}$$

   pour un certain $\alpha \in [0, 1]$.

Chapitre 2

The constraint equations in the asymptotically flat case

2.1 Introduction

Einstein equations can be formulated as a Cauchy problem whose initial data must satisfy compatibility conditions known as the constraint equations. In this paper, we will consider the constraint equations for the vacuum Einstein equations, in the particular case where the space-time possesses a space-like translational Killing field. It allows for a reduction of the $3+1$ dimensional problem to a $2+1$ dimensional one. This symmetry has been studied by Choquet-Bruhat and Moncrief in [11] (see also [8]) in the case of a space-time of the form $\Sigma \times \mathbb{S}^1 \times \mathbb{R}$, where Σ is a compact two dimensional manifold of genus $G \geq 2$, and $\mathbb{R}$ is the time axis, with a space-time metric independent of the $\mathbb{S}^1$ coordinate. They prove the existence of global solutions corresponding to perturbation of particular expanding initial data.

In this chapter we consider a space-time of the form $\mathbb{R}^2 \times \mathbb{R}_{x_3} \times \mathbb{R}_t$, symmetric with respect to the third coordinate. Minkowski space-time is a particular solution of vacuum Einstein equations which exhibits this symmetry. Since the celebrated work of Christodoulou and Klainerman (see [13]), we know that Minkowski space-time is stable, that is to say asymptotically flat perturbations of the trivial initial data lead to global solutions converging to Minkowski space-time. It is an interesting problem to ask whether the stability also holds in the setting of perturbations of Minkowski space-time with a space-like translational Killing field. Let's note that it is not included in the work of Christodoulou and Klainerman. However, it is crucial, before considering this problem, to ensure the existence of compatible initial data. In [29], we proved the existence of solutions to the constraint equations. The purpose of this chapter is to go further in the asymptotic development of the solutions to the constraint equations.

In the compact case, if one looks for solutions with constant mean curvature, as it is done in [11], the issue of solving the constraint equations is straightforward. Every metric on a compact manifold of genus $G \geq 2$ is conformal to a metric of scalar curvature -1. As a consequence, it is possible to decouple the system into elliptic scalar equations of the form $\Delta u = f(x,u)$ with $\partial_u f > 0$, for which existence results are standard (see for example chapter 14 in [46]).

The asymptotically flat case is more challenging. First, the definition of an asymptotically flat manifold is not so clear in two dimension. In [5], [3], [6] radial solutions of the $2+1$ dimensional problem with an angle at space-like infinity are constructed. In particular, these solutions do not tend to the Euclidean metric at space-like infinity. Moreover, the behavior of the Laplace operator on $\mathbb{R}^2$ makes the issue of finding solutions to the constraint equations more intricate.

2.1.1 Reduction of the Einstein equations

Before discussing the constraint equations, we first briefly recall the form of the Einstein equations in the presence of a space-like translational Killing field. We follow here the exposition in [8] (see also Section (1.5.1.1) for more details on the reduction). A metric ${}^{(4)}\mathbf{g}$ on $\mathbb{R}^2 \times \mathbb{R} \times \mathbb{R}$ admitting ∂_3 as a Killing field can be written

$${}^{(4)}\mathbf{g} = \widetilde{\mathbf{g}} + e^{2\gamma}(dx^3 + A_\alpha dx^\alpha)^2,$$

where $\widetilde{\mathbf{g}}$ is a Lorentzian metric on $\mathbb{R}^{1+2}$, γ is a scalar function on $\mathbb{R}^{1+2}$, A is a 1-form on $\mathbb{R}^{1+2}$ and x^α, $\alpha = 0, 1, 2$, are the coordinates on $\mathbb{R}^{1+2}$. Since ∂_3 is a Killing field, $\mathbf{g}$, γ and A do not depend on x^3. We set $F = dA$, where d is the exterior differential. F is then a 2-form. Let also ${}^{(4)}\mathbf{R}_{\mu\nu}$ denote the Ricci tensor associated to ${}^{(4)}\mathbf{g}$. $\widetilde{\mathbf{R}}_{\alpha\beta}$ and $\widetilde{\mathbf{D}}$ are respectively the Ricci tensor and the covariant derivative associated to $\widetilde{\mathbf{g}}$.

With this metric, the vacuum Einstein equations

$${}^{(4)}\mathbf{R}_{\mu\nu} = 0, \ \mu, \nu = 0, 1, 2, 3$$

can be written in the basis $(dx^\alpha, dx^3 + A_\alpha dx^\alpha)$ (see [8] appendix VII)

$$0 = {}^{(4)}\mathbf{R}_{\alpha\beta} = \widetilde{\mathbf{R}}_{\alpha\beta} - \frac{1}{2}e^{2\gamma}F_\alpha{}^\lambda F_{\beta\lambda} - \widetilde{\mathbf{D}}_\alpha\partial_\beta\gamma - \partial_\alpha\gamma\partial_\beta\gamma, \tag{2.1.1}$$

$$0 = {}^{(4)}\mathbf{R}_{\alpha 3} = \frac{1}{2}e^{-\gamma}\widetilde{\mathbf{D}}_\beta(e^{3\gamma}F_\alpha{}^\beta), \tag{2.1.2}$$

$$0 = {}^{(4)}\mathbf{R}_{33} = -e^{-2\gamma}\left(-\frac{1}{4}e^{2\gamma}F_{\alpha\beta}F^{\alpha\beta} + \widetilde{\mathbf{g}}^{\alpha\beta}\partial_\alpha\gamma\partial_\beta\gamma + \widetilde{\mathbf{g}}^{\alpha\beta}\widetilde{\mathbf{D}}_\alpha\partial_\beta\gamma\right). \tag{2.1.3}$$

The equation (2.1.2) is equivalent to

$$d(*e^{3\gamma}F) = 0$$

where $*e^{3\gamma}F$ is the adjoint one form associated to $e^{3\gamma}F$. This is equivalent, on $\mathbb{R}^{1+2}$, to the existence of a potential ω such that

$$*e^{3\gamma}F = d\omega.$$

Since F is a closed 2-form, we have $dF = 0$. By doing the conformal change of metric $\widetilde{\mathbf{g}} = e^{-2\gamma}\mathbf{g}$, this equation, together with the equations (2.1.1) and (2.1.3), yield the following system,

$$\Box_{\mathbf{g}}\omega - 4\partial^\alpha\gamma\partial_\alpha\omega = 0, \tag{2.1.4}$$

$$\Box_{\mathbf{g}}\gamma + \frac{1}{2}e^{-4\gamma}\partial^\alpha\omega\partial_\alpha\omega = 0, \tag{2.1.5}$$

$$\mathbf{R}_{\alpha\beta} = 2\partial_\alpha\gamma\partial_\beta\gamma + \frac{1}{2}e^{-4\gamma}\partial_\alpha\omega\partial_\beta\omega, \ \alpha, \beta = 0, 1, 2, \tag{2.1.6}$$

where $\Box_{\mathbf{g}}$ is the d'Alembertian[1] in the metric $\mathbf{g}$ and $\mathbf{R}_{\alpha\beta}$ is the Ricci tensor associated to $\mathbf{g}$. We introduce the following notation

$$u \equiv (\gamma, \omega), \tag{2.1.7}$$

together with the scalar product

$$\partial_\alpha u.\partial_\beta u = 2\partial_\alpha\gamma\partial_\beta\gamma + \frac{1}{2}e^{-4\gamma}\partial_\alpha\omega\partial_\beta\omega. \tag{2.1.8}$$

We consider the Cauchy problem for the equations (2.1.4), (2.1.5) and (2.1.6). As it is in the case for the $3+1$ Einstein equation, the initial data for (2.1.4), (2.1.5) and (2.1.6) cannot be prescribed arbitrarily. They have to satisfy constraint equations.

[1] $\Box_{\mathbf{g}}$ is the Lorentzian equivalent of the Laplace-Beltrami operator in Riemannian geometry. In a coordinate system, we have $\Box_{\mathbf{g}}u = \frac{1}{\sqrt{|\mathbf{g}|}}\partial_\alpha(\mathbf{g}^{\alpha\beta}\sqrt{|\mathbf{g}|}\partial_\beta u)$.

2.1.2 Constraint equations

We can write the metric $\mathbf{g}$ under the form

$$\mathbf{g} = -N^2(dt)^2 + g_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt), \tag{2.1.9}$$

where the scalar function N is called the lapse, the vector field β is called the shift and g is a Riemannian metric on $\mathbb{R}^2$.

We consider the initial space-like surface $\mathbb{R}^2 = \{t = 0\}$. Let T be the unit normal to $\mathbb{R}^2 = \{t = 0\}$. We set

$$e_0 = NT = \partial_t - \beta^j \partial_j.$$

We will use the notation

$$\partial_0 = \mathcal{L}_{e_0} = \partial_t - \mathcal{L}_\beta,$$

where $\mathcal{L}$ is the Lie derivative. With this notation, we have the following expression for the second fundamental form of $\mathbb{R}^2$

$$K_{ij} = -\frac{1}{2N}\partial_0 g_{ij}.$$

We will use the notation

$$\tau = g^{ij} K_{ij}$$

for the mean curvature. We also introduce the Einstein tensor

$$\mathbf{G}_{\alpha\beta} = \mathbf{R}_{\alpha\beta} - \frac{1}{2}\mathbf{R}\mathbf{g}_{\alpha\beta},$$

where $\mathbf{R}$ is the scalar curvature $\mathbf{R} = \mathbf{g}^{\alpha\beta}\mathbf{R}_{\alpha\beta}$. The constraint equations are given by

$$\mathbf{G}_{0j} \equiv N(\partial_j \tau - D^i K_{ij}) = \partial_0 u.\partial_j u, \; j = 1, 2, \tag{2.1.10}$$

$$\mathbf{G}_{00} \equiv \frac{N^2}{2}(R - |K|^2 + \tau^2) = \partial_0 u.\partial_0 u - \frac{1}{2}\mathbf{g}_{00}\mathbf{g}^{\alpha\beta}\partial_\alpha u \partial_\beta u, \tag{2.1.11}$$

where D and R are respectively the covariant derivative and the scalar curvature associated to g (see [8] chapter VI for a derivation of (2.1.10) and (2.1.11)). Equation (2.1.10) is called the momentum constraint and (2.1.11) is called the Hamiltonian constraint. If we came back to the $3+1$ problem, there should be four constraint equations. However, since the fourth would be obtained by taking $\alpha = 0$ in (2.1.2), it is trivially satisfied if we set $*e^{3\gamma}F = d\omega$.

We will look for g of the form $g = e^{2\lambda}\delta$ where δ is the Euclidean metric on $\mathbb{R}^2$. There is no loss of generality since, up to a diffeomorphism, all metrics on $\mathbb{R}^2$ are conformal to the Euclidean metric. We introduce the traceless part of K,

$$H_{ij} = K_{ij} - \frac{1}{2}\tau g_{ij},$$

and following [11] we introduce the quantity

$$\dot{u} = \frac{e^{2\lambda}}{N}\partial_0 u.$$

Then the equations (2.1.10) and (2.1.11) take the form

$$\partial_i H_{ij} = -\dot{u}.\partial_j u + \frac{1}{2}e^{2\lambda}\partial_j \tau, \tag{2.1.12}$$

$$\Delta\lambda + e^{-2\lambda}\left(\frac{1}{2}\dot{u}^2 + \frac{1}{2}|H|^2\right) - e^{2\lambda}\frac{\tau^2}{4} + \frac{1}{2}|\nabla u|^2 = 0, \tag{2.1.13}$$

where here and in the remaining of the paper, we use the convention for the Laplace operator

$$\Delta = \partial_1^2 + \partial_2^2.$$

The aim of this chapter is to solve the coupled system of nonlinear elliptic equations (2.1.12) and (2.1.13) on $\mathbb{R}^2$ in the small data case, that is to say when $\dot{u}$ and ∇u are small. A similar system can be obtained when studying the constraint equations in three dimensions by using the conformal method, introduced by Lichnerowicz [37] and Choquet-Bruhat and York [12]. In the constant mean curvature (CMC) case, that is to say when one sets $\tau = 0$, the constraint equations decouple and the main difficulty that remains is the study of the scalar equation (2.1.13), also called the Lichnerowicz equation[2]. The CMC solutions have been studied in [12] and [30] for the compact case, and in [7] for the asymptotically flat case. There have been also some results concerning the coupled constraint equations, i.e. without setting τ constant The near CMC solutions in the asymptotically flat case have been studied in [10]. The compact case has been studied in [27], [42] and [16]. See also [4] for a review of these results.

In our case, the difficulty will arise from particular issues concerning the inversion of second order elliptic operators on $\mathbb{R}^2$. In particular, without special assumptions on u, it is not possible to set $\tau = 0$ in the case of $\mathbb{R}^2$. Indeed, equation (2.1.12) induces for H the asymptotic $|H|^2 \sim \frac{1}{r^2}$ as r tends to infinity. Now, it is known (see [45]) that an equation of the form

$$\Delta u + Re^{2u} + f = 0,$$

with $R, f \leq 0$ and $R \lesssim -\frac{1}{r^2}$ when r tends to infinity, admits no solution. Therefore, we will be forced to carefully adjust the asymptotic behavior of τ as r tends to infinity, to compensate the term $|H|^2$ in equation (2.1.13), and to ensure that we remain in the range of the elliptic operators which come into play.

Remark 2.1.1. *The solution of equation* (2.1.13) *that we construct in this paper satisfies*

$$\lambda = -\alpha \ln(r) + o(1), \tag{2.1.14}$$

as $r \to \infty$, with $\alpha > 0$. At first sight, this could seem to contradict the asymptotic flatness we are looking for. However, we mentioned in the beginning of the introduction that it is not so clear what to expect as a definition of asymptotic flatness in $2+1$ dimension. The solutions of the evolution problem (2.1.4), (2.1.5) *and* (2.1.6) *with an additional rotational symmetry and $\omega \equiv 0$, known as Einstein-Rosen waves, have been studied in [5] and [3]. These solutions exhibit a conical singularity at space-like infinity, that is to say the perimeter of a circle of radius r asymptotically grows like $2\pi c r$ with $c < 1$, instead of $2\pi r$ in the Euclidean metric.*

Using a change of variable, we observe that the asymptotic behavior (2.1.14) *is equivalent to the presence of an asymptotic angle at space-like infinity. Indeed, if we make the change of coordinate $r' = \frac{r^{1-\alpha}}{1-\alpha}$ for r large enough, then the metric*

$$g \sim r^{-2\alpha}(dr^2 + r^2 d\theta^2), \quad r \to \infty$$

takes the form

$$g' \sim dr'^2 + (1-\alpha)^2 r'^2 d\theta^2, \quad r' \to \infty$$

which corresponds to a conical singularity at space-like infinity, with an angle given by

$$2\pi(1-\alpha).$$

Note that, since the constraint equations (2.1.10) *and* (2.1.11) *are independent of the choice of coordinates, the metric g' and the second fundamental form K', obtained by performing the change of variables $r' = \frac{r^{1-\alpha}}{1-\alpha}$ for r large enough, are still solutions of the constraint equations.*

[2]The resolution of this equation is closely linked to the Yamabe problem

We will do the following rescaling to avoid the $e^{2\lambda}$ and $e^{-2\lambda}$ factors

$$\breve{u} = e^{-\lambda}\dot{u}, \quad \breve{H} = e^{-\lambda}H, \quad \breve{\tau} = e^{\lambda}\tau.$$

Then the equations (2.1.12) and (2.1.13) become

$$\begin{aligned} &\partial_i \breve{H}_{ij} + \breve{H}_{ij}\partial_i\lambda = -\breve{u}.\partial_j u + \frac{1}{2}\partial_j\breve{\tau} - \frac{1}{2}\breve{\tau}\partial_j\lambda, \\ &\Delta\lambda + \frac{1}{2}\breve{u}^2 + \frac{1}{2}|\nabla u|^2 + \frac{1}{2}|\breve{H}|^2 - \frac{\breve{\tau}^2}{4} = 0. \end{aligned}$$

To lighten the notations, we will omit the $\breve{}$ in the rest of the paper. We consider therefore the system

$$\begin{cases} \partial_i H_{ij} + H_{ij}\partial_i\lambda = -\dot{u}.\partial_j u + \frac{1}{2}\partial_j\tau - \frac{1}{2}\tau\partial_j\lambda, \\ \Delta\lambda + \frac{1}{2}\dot{u}^2 + \frac{1}{2}|\nabla u|^2 + \frac{1}{2}|H|^2 - \frac{\tau^2}{4} = 0. \end{cases} \tag{2.1.15}$$

Before stating the main result, we recall several properties of weighted Sobolev spaces.

2.2 Preliminaries

2.2.1 Weighted Sobolev spaces

In the rest of the paper, $\chi(r)$ denotes a smooth non negative function such that

$$0 \le \chi \le 1, \quad \chi(r) = 0 \text{ for } r \le 1, \quad \chi(r) = 1 \text{ for } r \ge 2.$$

We will also note $f \lesssim h$ when there exists a universal constant C such that $f \le Ch$.

Definition 2.2.1. *Let $m \in \mathbb{N}$ and $\delta \in \mathbb{R}$. The weighted Sobolev space $H^m_\delta(\mathbb{R}^n)$ is the completion of C^∞_0 for the norm*

$$\|u\|_{H^m_\delta} = \sum_{|\beta|\le m} \|(1+|x|^2)^{\frac{\delta+|\beta|}{2}} D^\beta u\|_{L^2}.$$

The weighted Hölder space C^m_δ is the complete space of m-times continuously differentiable functions with norm

$$\|u\|_{C^m_\delta} = \sum_{|\beta|\le m} \|(1+|x|^2)^{\frac{\delta+|\beta|}{2}} D^\beta u\|_{L^\infty}.$$

Let $0<\alpha<1$. The Hölder space $C^{m+\alpha}_\delta$ is the the complete space of m-times continuously differentiable functions with norm

$$\|u\|_{C^{m+\alpha}_\delta} = \|u\|_{C^m_\delta} + \sup_{x\neq y,\ |x-y|\le 1} \frac{|\partial^m u(x) - \partial^m u(y)|(1+|x|^2)^{\frac{\delta}{2}}}{|x-y|^\alpha}.$$

The following lemma is an immediate consequence of the definition.

Lemma 2.2.2. *Let $m \ge 1$ and $\delta \in \mathbb{R}$. Then $u \in H^m_\delta$ implies $\partial_j u \in H^{m-1}_{\delta+1}$ for $j = 1, .., n$.*

We first recall the Sobolev embedding with weights (see for example [8], Appendix I). In the rest of this section, we assume $n = 2$.

Proposition 2.2.3. *Let $s, m \in \mathbb{N}$. We assume $s > 1$. Let $\beta \le \delta+1$ and $0 < \alpha < min(1, s-1)$. Then, we have the continuous embedding*

$$H^{s+m}_\delta \subset C^{m+\alpha}_\beta.$$

We will also need a product rule.

Proposition 2.2.4. *Let $s, s_1, s_2 \in \mathbb{N}$. We assume $s \leq \min(s_1, s_2)$ and $s < s_1+s_2-1$. Let $\delta < \delta_1+\delta_2+1$. Then $\forall (u,v) \in H^{s_1}_{\delta_1} \times H^{s_2}_{\delta_2}$,*
$$\|uv\|_{H^s_\delta} \lesssim \|u\|_{H^{s_1}_{\delta_1}} \|v\|_{H^{s_2}_{\delta_2}}.$$

The following simple lemma will be useful as well.

Lemma 2.2.5. *Let $\alpha \in \mathbb{R}$ and $g \in L^\infty_{loc}$ be such that*
$$|g(x)| \lesssim (1+|x|^2)^\alpha.$$
Then the multiplication by g maps H^0_δ to $H^0_{\delta-2\alpha}$.

We will also need the following modified version of Lemma (2.2.5).

Lemma 2.2.6. *Let $\alpha \in \mathbb{R}$ and $g_1 \in L^\infty_{loc}$ be a function such that*
$$|g_1(x)| \lesssim (1+|x|^2)^\alpha.$$
Let $g_2 \in L^2(\mathbb{S}^1)$. Then the multiplication by $g_1(x)g_2(\theta)$ maps H^1_δ to $H^0_{\delta-2\alpha}$.

Proof. Let $u \in H^1_\delta$. We estimate
$$\begin{aligned}
&\int_0^\infty \int_0^{2\pi} (1+r^2)^{\delta-2\alpha} g_1(x)^2 g_2(\theta)^2 u(r,\theta)^2 r dr d\theta \\
&\leq \|g_2\|^2_{L^2(\mathbb{S}^1)} \int_0^\infty (1+r^2)^{\delta-2\alpha} \left(\sup_{\theta\in[0,2\pi]} |g_1|(r,\theta)\right)^2 \left(\sup_{\theta\in[0,2\pi]} |u|(r,\theta)\right)^2 r dr \\
&\lesssim \|g_2\|^2_{L^2(\mathbb{S}^1)} \int_0^\infty (1+r^2)^{\delta} \left(\int_0^{2\pi} |u|^2 + |\partial_\theta u|^2 d\theta\right) r dr \\
&\lesssim \|g_2\|^2_{L^2(\mathbb{S}^1)} \left(\int (1+r^2)^\delta u^2 dx + \int (1+r^2)^{\delta+1} |\nabla u|^2 dx\right) \\
&\lesssim \|g_2\|^2_{L^2(\mathbb{S}^1)} \|u\|^2_{H^1_\delta}
\end{aligned}$$
where we have used the Sobolev embedding of $L^\infty(\mathbb{S}^1)$ in the Sobolev space $W^{1,2}(\mathbb{S}^1)$. □

We will use the following definition

Definition 2.2.7. *Let $\delta \in \mathbb{R}$ and $s \in \mathbb{N}$. We note $\mathcal{H}^s_\delta$ the set of symmetric traceless 2-tensors whose components are in H^s_δ.*

2.2.2 Behavior of the Laplace operator in weighted Sobolev spaces.

Theorem 2.2.8. *(Theorem 0 in [43]) Let $m \in \mathbb{N}$ and $-1+m < \delta < m$. The Laplace operator $\Delta : H^2_\delta \to H^0_{\delta+2}$ is an injection with closed range*
$$\left\{ f \in H^0_{\delta+2} \mid \int fv = 0 \quad \forall v \in \cup_{i=0}^m \mathcal{H}_i \right\},$$
where $\mathcal{H}_i$ is the set of harmonic polynomials of degree i. Moreover, u obeys the estimate
$$\|u\|_{H^2_\delta} \leq C(\delta) \|\Delta u\|_{H^0_{\delta+2}},$$
where $C(\delta)$ is a constant such that $C(\delta) \to +\infty$ when $\delta \to m_-$ and $\delta \to (-1+m)_+$.

We will prove three corollaries of Theorem (2.2.8) which will be fundamental in our work.

Corollary 2.2.9. *Let $s, m \in \mathbb{N}$ and $-1+m<\delta<m$. The Laplace operator $\Delta : H^{2+s}_\delta \to H^s_{\delta+2}$ is an injection with closed range*

$$\left\{ f \in H^s_{\delta+2} \mid \int fv = 0 \quad \forall v \in \cup_{i=0}^m \mathcal{H}_i \right\}.$$

Moreover, u obeys the estimate

$$\|u\|_{H^{s+2}_\delta} \leq C(s,\delta)\|\Delta u\|_{H^s_{\delta+2}}.$$

Proof. We will proceed by induction. Note that Theorem (2.2.8) corresponds to the case $s=0$. We assume that the statement of the corollary holds true for some $s \in \mathbb{N}$ and all $m \in \mathbb{N}$, and we will prove that it holds true for $s+1$. Let $m \in \mathbb{N}$ and $-1+m<\delta<m$. Let $f \in H^{s+1}_{\delta+2}$, such that f belongs to the set

$$\left\{ f \in H^0_{\delta+2} \mid \int fv = 0 \quad \forall v \in \cup_{i=0}^m \mathcal{H}_i \right\}.$$

Then Theorem (2.2.8) provides a unique $u \in H^2_\delta$ such that $\Delta u = f$. In particular for $i=1,2$ we have

$$\Delta \partial_i u = \partial_i f.$$

Since $f \in H^{s+1}_{\delta+2}$, we have that $\partial_i f \in H^s_{\delta+3}$. Moreover, for all v, harmonic polynomial of degree $j \leq m+1$, we have

$$\int (\partial_i f) v = -\int f \partial_i v = 0,$$

because $\partial_i v$ is an harmonic polynomial of degree $j-1 \leq m$. Therefore, by induction, we have $\partial_i u \in H^{s+2}_{\delta+1}$ and

$$\begin{aligned}
\|u\|_{H^{s+1+2}_\delta} &\lesssim \|u\|_{H^2_\delta} + \|\partial_1 u\|_{H^{s+2}_{\delta+1}} + \|\partial_2 u\|_{H^{s+2}_{\delta+1}} \\
&\leq C(\delta)\|f\|_{H^0_{\delta+2}} + C(s,\delta+1)\left(\|\partial_1 f\|_{H^s_{\delta+3}} + \|\partial_2 f\|_{H^s_{\delta+3}} \right) \\
&\leq C(s+1,\delta)\|f\|_{H^{s+1}_{\delta+2}}.
\end{aligned}$$

This concludes the proof of Corollary (2.2.9). ☐

Corollary 2.2.10. *Let $-1<\delta<0$. Let $f \in H^0_{\delta+3}$. Then there exists a solution u of*

$$\Delta u = f,$$

which can be written uniquely in the form

$$u = \frac{1}{2\pi}\left(\int f\right)\chi(r)\ln(r) - \frac{1}{2\pi}\left(\cos(\theta)\int f x_1 + \sin(\theta)\int f x_2\right)\frac{\chi(r)}{r} + \widetilde{u},$$

where $\widetilde{u} \in H^2_{\delta+1}$. Moreover, we have the estimate

$$\|\tilde{u}\|_{H^2_{\delta+1}} \lesssim C(\delta)\|f\|_{H^0_{\delta+3}}.$$

Proof. Let F be a radial function, smooth, compactly supported, such that $\int F = 2\pi$, and G a radial function, smooth, compactly supported, which is 0 in a neighborhood of 0 and such that $\int Gr = 4\pi$. We note

$$G_1(x) = G(r)\cos(\theta) \text{ and } G_2(x) = G(r)\sin(\theta).$$

Let

$$u_0(x) = \frac{1}{2\pi}\int F(y)\ln(|x-y|)dy$$

be a solution of $\Delta u_0 = F$, and

$$u_i(x) = \frac{1}{2\pi}\int G_i(y)\ln(|x-y|)dy$$

be a solution of $\Delta u_i = G_i$. We may calculate

$$\begin{aligned} u_0 &= \chi(r)\ln(r) + \widetilde{u}_0, \\ u_1 &= -\chi(r)\frac{\cos(\theta)}{r} + \widetilde{u}_1, \\ u_2 &= -\chi(r)\frac{\sin(\theta)}{r} + \widetilde{u}_2, \end{aligned}$$

where $\widetilde{u}_0, \widetilde{u}_i \in H^2_{\delta+1}$.

Thanks to Theorem (2.2.8), we can solve the equation

$$\Delta v = f - \frac{1}{2\pi}\left(\int f\right)F - \frac{1}{2\pi}\left(\int fx_1\right)G_1 - \frac{1}{2\pi}\left(\int fx_2\right)G_2$$

since the right-hand side is orthogonal to the polynomials of degree 0 and 1, and we have $v \in H^2_{\delta+1}$, which satisfies

$$\|v\|_{H^2_{\delta+1}} \lesssim \|f\|_{H^0_{\delta+3}} + \int |f| + \int r|f| \lesssim \|f\|_{H^0_{\delta+3}} + \int |f|\frac{(1+r^2)^{\frac{\delta}{2}+\frac{3}{2}}}{(1+r^2)^{\frac{\delta}{2}+1}} \lesssim \frac{1}{\sqrt{\delta+1}}\|f\|_{H^0_{\delta+3}}.$$

Therefore we can solve the equation $\Delta u = f$, and u can be written

$$\begin{aligned} u &= v + \frac{1}{2\pi}\left(\int f\right) + \frac{1}{2\pi}\left(\int fx_1\right)u_1 + \frac{1}{2\pi}\left(\int fx_2\right)u_2 \\ &= \frac{1}{2\pi}\left(\int f\right)\chi(r)\ln(r) - \frac{1}{2\pi}\left(\cos(\theta)\int fx_1 + \sin(\theta)\int fx_2\right)\frac{\chi(r)}{r} + \widetilde{u}, \end{aligned}$$

where $\widetilde{u} \in H^2_{\delta+1}$ with

$$\|\tilde{u}\|_{H^2_{\delta+1}} \lesssim \|f\|_{H^0_{\delta+3}}.$$

This concludes the proof of Corollary 2.2.10. □

We introduce the notation.

$$M_\theta = \begin{pmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{pmatrix}, \quad N_\theta = \begin{pmatrix} -\sin(2\theta) & \cos(2\theta) \\ \cos(2\theta) & \sin(2\theta) \end{pmatrix}.$$

Corollary 2.2.11. *Let $-1 < \delta < 0$. Let $f_j \in H^0_{\delta+3}$ with $\int f_j = 0$, $j = 1, 2$. Then, there exists a symmetric and traceless 2-tensor K solution of*

$$\partial_i K_{ij} = f_j,$$

which can be written uniquely in the form

$$K = A\frac{\chi(r)}{r^2}M_\theta + J\frac{\chi(r)}{r^2}N_\theta + \widetilde{K},$$

with $\widetilde{K} \in \mathcal{H}^1_{\delta+2}$ and

$$A = \frac{1}{2\pi}\int x_1f_1 + x_2f_2, \quad J = \frac{1}{2\pi}\int x_1f_2 - x_2f_1, \quad \|\widetilde{K}\|_{\mathcal{H}^1_{\delta+2}} \lesssim \|f_1\|_{H^0_{\delta+3}} + \|f_1\|_{H^0_{\delta+3}}.$$

Proof. We can look for K of the form

$$K_{ij} = \partial_i Y_j + \partial_j Y_i - \delta_{ij}\partial^k Y_k,$$

then Y_j satisfies

$$\Delta Y_j = f_j.$$

We apply Corollary (2.2.10) which allows us to find a solution in the form[3]

$$Y_j = \frac{\chi(r)}{r}\left(a_j\cos(\theta) + b_j\sin(\theta)\right) + \widetilde{Y}_j,$$

with $\widetilde{Y}_j \in H^2_{\delta+1}$ and

$$a_j = -\frac{1}{2\pi}\int x_1 f_j, \quad b_j = -\frac{1}{2\pi}\int x_2 f_j, \quad \|\widetilde{Y}_j\|_{H^2_{\delta+1}} \lesssim \|f_j\|_{H^0_{\delta+3}}.$$

We calculate

$$\begin{aligned}
K_{11} =& \partial_1 Y_1 - \partial_2 Y_2\\
=& \frac{\chi(r)}{r^2}\Big(a_1(-\cos^2(\theta) + \sin^2(\theta)) - 2b_1\cos(\theta)\sin(\theta) + 2a_2\cos(\theta)\sin(\theta)\\
&+ b_2(\sin^2(\theta) - \cos^2(\theta))\Big) + \widetilde{K}_{11}\\
=& \frac{\chi(r)}{r^2}\left(-(a_1+b_2)\cos(2\theta) + (a_2 - b_1)\sin(2\theta)\right) + \widetilde{K}_{11},\\
K_{12} =& \partial_1 Y_2 + \partial_2 Y_1\\
=& \frac{\chi(r)}{r^2}\Big(a_2(-\cos^2(\theta) + \sin^2(\theta)) - 2b_2\cos(\theta)\sin(\theta) - 2a_1\cos(\theta)\sin(\theta)\\
&+ b_1(-\sin^2(\theta) + \cos^2(\theta))\Big) + \widetilde{K}_{12}\\
=& \frac{\chi(r)}{r^2}\left(-(a_2-b_1)\cos(2\theta) - (a_1 + b_2)\sin(2\theta)\right) + \widetilde{K}_{12},
\end{aligned}$$

with $\widetilde{K}_{11}$ and $\widetilde{K}_{12}$ in $H^1_{\delta+2}$ and

$$\|\widetilde{K}_{11}\|_{H^1_{\delta+2}} + \|\widetilde{K}_{12}\|_{H^1_{\delta+2}} \lesssim \|f_1\|_{H^0_{\delta+3}} + \|f_2\|_{H^0_{\delta+3}}.$$

Therefore we can write

$$K = A\frac{\chi(r)}{r^2}\begin{pmatrix}\cos(2\theta) & \sin(2\theta)\\ \sin(2\theta) & -\cos(2\theta)\end{pmatrix} + J\frac{\chi(r)}{r^2}\begin{pmatrix}-\sin(2\theta) & \cos(2\theta)\\ \cos(2\theta) & \sin(2\theta)\end{pmatrix} + \widetilde{K},$$

with

$$\begin{aligned}
A = -(a_1 + b_2) &= \frac{1}{2\pi}\int x_1 f_1 + x_2 f_2,\\
J = b_1 - a_2 &= \frac{1}{2\pi}\int x_1 f_2 - x_2 f_1,\\
\|\widetilde{K}\|_{\mathcal{H}^1_{\delta+2}} &\lesssim \|f_1\|_{H^0_{\delta+2}} + \|f_2\|_{H^0_{\delta+2}}.
\end{aligned}$$

This concludes the proof of Corollary 2.2.11. □

[3] Recall that $\int f_j = 0$.

2.3 Main result and outline of the proof

In [29], we solved the system (2.1.15) for $\dot{u}^2, |\nabla u|^2 \in H^0_{\delta+2}$ with $-1 < \delta < 0$. The solutions we found were of the form

$$\begin{aligned}
\lambda &= -\alpha\chi(r)\ln(r) + \widetilde{\lambda}, \\
H &= -(\rho\cos(\theta-\eta) + \widetilde{b}(\theta))\frac{\chi(r)}{2r}M_\theta + \widetilde{H}, \\
\tau &= (\rho\cos(\theta-\eta) + \widetilde{b}(\theta))\frac{\chi(r)}{r} + \widetilde{\tau},
\end{aligned}$$

where $\widetilde{\lambda} \in H^2_\delta$, $\widetilde{H} \in \mathcal{H}^1_{\delta+1}$. By looking for H as $H_{ij} = \partial_i Y_j + \partial_j Y_i - \delta_{ij}\partial^k Y_k$, the system (2.1.15) corresponds to three Laplace-like equations. The quantities $\widetilde{b} \in W^{1,2}(\mathbb{S}^1)$ and $\widetilde{\tau} \in H^1_{\delta+1}$ are free parameters, while the three parameters α, ρ and η are determined by the three corresponding orthogonality conditions, namely that the integrals of the right-hand sides of (2.1.15) vanish.

In this chapter, assuming that $\dot{u}^2, |\nabla u|^2 \in H^0_{\delta+3}$ (i.e. assuming more decay on u and $\dot{u}$ than in [29]), we want to go further in the asymptotic expension of our solution. This will require to enforce additional orthogonality conditions.

2.3.1 Main result

Theorem 2.3.1. *Let $-1 < \delta < 0$. Let $\dot{u}^2, |\nabla u|^2 \in H^0_{\delta+3}$ and $\widetilde{b} \in W^{1,2}(\mathbb{S}^1)$ such that*

$$\int_{\mathbb{S}^1} \widetilde{b}(\theta)\cos(\theta)d\theta = \int_{\mathbb{S}^1} \widetilde{b}(\theta)\sin(\theta)d\theta = 0. \tag{2.3.1}$$

We note

$$\varepsilon = \int \dot{u}^2 + |\nabla u|^2.$$

We assume

$$\|\dot{u}^2\|_{H^0_{\delta+3}} + \||\nabla u|^2\|_{H^0_{\delta+3}} + \|\widetilde{b}\|_{W^{1,2}} \lesssim \varepsilon.$$

Let $B \in W^{1,2}(\mathbb{S}^1)$. We assume

$$\|B\|_{W^{1,2}} \lesssim \varepsilon^2.$$

Let $\Psi \in H^1_{\delta+2}$ be such that $\int \Psi = 2\pi$. If $\varepsilon > 0$ is small enough, there exist $\alpha, \rho, \eta, A, J, c_1, c_2$ in $\mathbb{R}$, a scalar functions $\widetilde{\lambda} \in H^2_{\delta+1}$ and a symmetric traceless tensor $\widetilde{H} \in \mathcal{H}^1_{\delta+2}$ such that, if r, θ are the polar coordinates centered in (c_1, c_2), and if we note

$$\begin{aligned}
\lambda &= -\alpha\chi(r)\ln(r) + \widetilde{\lambda}, \\
H &= -(\widetilde{b}(\theta) + \rho\cos(\theta-\eta))\frac{\chi(r)}{2r}M_\theta + e^{-\lambda}\frac{\chi(r)}{r^2}\left((J-(1-\alpha)B(\theta))N_\theta - \frac{B'(\theta)}{2}M_\theta\right) + \widetilde{H},
\end{aligned}$$

then λ, H are solutions of (2.1.15) *with*

$$\tau = (\widetilde{b}(\theta) + \rho\cos(\theta-\eta))\frac{\chi(r)}{r} + e^{-\lambda}B'(\theta)\frac{\chi(r)}{r^2} + A\Psi.$$

Moreover we have the estimates

$$\begin{aligned}
\alpha &= \frac{1}{4\pi}\int \left(\dot{u}^2 + |\nabla u|^2\right) + O(\varepsilon^2),\\
\rho\cos(\eta) &= \frac{1}{\pi}\int \dot{u}.\partial_1 u + O(\varepsilon^2),\\
\rho\sin(\eta) &= \frac{1}{\pi}\int \dot{u}.\partial_2 u + O(\varepsilon^2),\\
c_1 &= \frac{1}{4\pi}\int x_1\left(\dot{u}^2 + |\nabla u|^2\right) + O(\varepsilon^2),\\
c_2 &= \frac{1}{4\pi}\int x_2\left(\dot{u}^2 + |\nabla u|^2\right) + O(\varepsilon^2),\\
J &= -\frac{1}{2\pi}\int \dot{u}.\partial_\theta u + \frac{\rho}{2\alpha}(c_1\sin(\eta) - c_2\cos(\eta)) + O(\varepsilon^2),\\
A &= -\frac{1}{2\pi}\int r\dot{u}.\partial_r u + \frac{1}{2\pi}\left(\int \chi'(r)rdr\right)\int \widetilde{b}(\theta)d\theta + O(\varepsilon^2),
\end{aligned}$$

and

$$\|\widetilde{\lambda}\|_{H^2_{\delta+1}} + \|\widetilde{\tau}\|_{H^1_{\delta+2}} + \|\widetilde{H}\|_{\mathcal{H}^1_{\delta+2}} \lesssim \varepsilon.$$

Remark 2.3.2. *There is a natural rapprochement between the quantities $\alpha,\rho,\eta,c_1,c_2,J,A$ and the global charges in $3+1$ dimensions (such as the ADM mass, ADM momentum...). See for example [15] for a definition.*

The following corollary is a straightforward consequence of Theorem (2.3.1) and Corollary (2.2.9).

Corollary 2.3.3. *Let $\delta,\dot{u},\nabla u,\varepsilon,\widetilde{b},B$ and Ψ be as in the assumptions of Theorem 2.3.1. Moreover let $s\in\mathbb{N}$ and assume $\dot{u}^2,|\nabla u|^2\in H^s_{\delta+3}$, $B,\widetilde{b}\in W^{s+1,2}(\mathbb{S}^1)$ and $\Psi\in H^{s+1}_{\delta+2}$. Then the conclusion of Theorem* (2.3.1) *holds and we have furthermore $\widetilde{\lambda}\in H^{s+2}_{\delta+1}$, $\widetilde{H}\in\mathcal{H}^{s+1}_{\delta+2}$, with the estimates*

$$\|\widetilde{\lambda}\|_{H^{s+2}_{\delta+1}} + \|\widetilde{H}\|_{\mathcal{H}^{s+1}_{\delta+2}} \lesssim \|\dot{u}^2\|_{H^s_{\delta+3}} + \||\nabla u|^2\|_{H^s_{\delta+3}} + \|\widetilde{b}\|_{W^{s+1,2}} + \|B\|_{W^{s+1,2}}.$$

2.3.2 Outline of the proof

We will prove the theorem using a fixed point argument.

Construction of the map F We consider the map

$$\begin{aligned}
F:\mathbb{R}\times\mathbb{R}\times\mathbb{R}\times H^2_{\delta+1} &\to \mathbb{R}\times\mathbb{R}\times\mathbb{R}\times H^2_{\delta+1}\\
(\alpha,c_1,c_2,\widetilde{\lambda}) &\mapsto (\alpha',c_1',c_2',\widetilde{\lambda}')
\end{aligned}$$

where if we note

$$(c_1,c_2) = r_c(\cos(\theta_c),\sin(\theta_c)),\quad (c_1',c_2') = r_c'(\cos(\theta_c'),\sin(\theta_c'))$$

and

$$\begin{aligned}
\lambda &= -\alpha\chi(r)\ln(r) + r_c\cos(\theta-\theta_c)\frac{\chi(r)}{r} + \widetilde{\lambda}\\
\lambda' &= -\alpha'\chi(r)\ln(r) + r_c'\cos(\theta-\theta_c')\frac{\chi(r)}{r} + \widetilde{\lambda}',
\end{aligned}$$

then λ' is the solution of

$$\Delta\lambda' + \frac{1}{2}\dot{u}^2 + \frac{1}{2}|\nabla u|^2 + \frac{1}{2}|H|^2 - \frac{\tau^2}{4} = 0, \tag{2.3.2}$$

with
$$H = e^{-\lambda}H^{(1)} + H^{(2)} + e^{-\lambda}H^{(3)}, \tag{2.3.3}$$
where
$$H^{(2)} = -b(\theta)\frac{\chi(r)}{2r}M_\theta - \frac{r_c}{2\alpha}(b(\theta)\sin(\theta-\theta_c))'\frac{\chi(r)}{r^2}M_\theta - \frac{r_c}{\alpha}b(\theta)\sin(\theta-\theta_c)\frac{\chi(r)}{r^2}N_\theta, \tag{2.3.4}$$
$$H^{(3)} = \frac{\chi(r)}{r^2}\left(-(1-\alpha)B(\theta)N_\theta - \frac{B'(\theta)}{2}M_\theta\right), \tag{2.3.5}$$
and H satisfies
$$\partial_i H_{ij} + H_{ij}\partial_i\lambda = -\dot{u}.\partial_j u + \frac{1}{2}\partial_j\tau - \frac{1}{2}\tau\partial_j\lambda, \tag{2.3.6}$$
with
$$\tau = \tau^{(2)} + e^{-\lambda}\tau^{(3)} + A\Psi, \tag{2.3.7}$$
where
$$\tau^{(2)} = b(\theta)\frac{\chi(r)}{r} + \frac{r_c}{\alpha}(b(\theta)\sin(\theta-\theta_c))'\frac{\chi(r)}{r^2}, \tag{2.3.8}$$
$$\tau^{(3)} = B'(\theta)\frac{\chi(r)}{r^2}. \tag{2.3.9}$$
We have noted
$$b(\theta) = \rho\cos(\theta-\eta) + \widetilde{b}(\theta). \tag{2.3.10}$$
The parameters ρ, η and A are suitably chosen during the process.

Solving (2.3.6) We will show that $H^{(1)}$ satisfies
$$\partial_i H^{(1)}_{ij} = f^{(1)}_j$$
with $f^{(1)}_j \in H^0_{\delta+3}$. We will prove that we may choose ρ, η and A such that
$$\int f^{(1)}_1 = \int f^{(2)}_2 = \int x_1 f^{(1)}_1 + x_2 f^{(1)}_2 = 0.$$
Then we will show that $H^{(1)}$ can be written
$$H^{(1)} = J\frac{\chi(r)}{r^2}N_\theta + \widetilde{H}^{(1)},$$
with $\widetilde{H}^{(1)} \in H^1_{\delta+2}$.

Solving (2.3.2) We will show that
$$\frac{1}{2}|H|^2 - \frac{\tau^2}{4} \in H^0_{\delta+3}.$$
Then, it will be straightforward to solve (2.3.2) using Corollary (2.2.10). The solution we obtain is of the form
$$\lambda' = -\alpha'\chi(r)\ln(r) + r'_c\cos(\theta-\theta'_c)\frac{\chi(r)}{r} + \widetilde{\lambda}',$$
with $\widetilde{\lambda}' \in H^2_{\delta+1}$.

The fixed point Proving that F is a contracting map easily follows from the estimates for λ' and H. The Picard fixed point theorem then implies that F has a fixed point. To obtain the result stated in Theorem (2.3.1) then easily folows after performing the following change of variables

$$x_1' = x_1 - \frac{r_c}{\alpha}\cos(\theta_c), \quad x_2' = x_2 + \frac{r_c}{\alpha}\sin(\theta_c),$$

which corresponds to work in a frame centered in the center of mass.

The rest of the paper is as follows. In section (2.4), we explain how to solve the momentum constraint (2.3.6). We also explain how to choose A, ρ, η. In section (2.5), we explain how to solve (2.3.2). Finally, the map F is shown to have a fixed point in section (2.6).

2.4 The momentum constraint

The goal of this section is to solve equation (2.3.6). We will note

$$\|\lambda\| = |\alpha| + r_c + \|\widetilde{\lambda}\|_{H^2_{\delta+1}}. \tag{2.4.1}$$

We assume a priori

$$\|\lambda\| \lesssim \varepsilon,$$
$$\alpha \geq \frac{1}{8\pi}\left(\int \dot{u}^2 + |\nabla u|^2\right). \tag{2.4.2}$$

This yields

$$\frac{r_c}{\alpha} \lesssim \frac{\|\lambda\|}{\varepsilon} \lesssim 1.$$

Proposition 2.4.1. *If $\varepsilon > 0$ is small enough, there exists $\rho, \eta, A \in \mathbb{R}$, such that for*

$$\tau = \tau^{(2)} + e^{-\lambda}\tau^{(3)} + A\Psi,$$

with $\tau^{(2)}, \tau^{(3)}$ defined by (2.3.8) and (2.3.9), there exists a solution of (2.3.6) which may be uniquely written under the form

$$H = e^{-\lambda}H^{(1)} + H^{(2)} + e^{-\lambda}H^{(3)}$$

where $H^{(2)}$ and $H^{(3)}$ are defined by (2.3.4) and (2.3.5) and

$$H^{(1)} = J\frac{\chi(r)}{r^2}N_\theta + \widetilde{H}^{(1)},$$

with $e^{-\lambda}\widetilde{H}^{(1)} \in \mathcal{H}^1_{\delta+2}$ such that

$$\|e^{-\lambda}\widetilde{H}^{(1)}\|_{\mathcal{H}^1_{\delta+2}} \lesssim \|\dot{u}\nabla u\|_{H^0_{\delta+3}} + \|b\|_{W^{1,2}} + \|B\|_{W^{1,2}} + |A| \lesssim \varepsilon.$$

Moreover we have the estimates

$$\begin{aligned}
\rho\cos(\eta) &= \frac{1}{\pi}\int \dot{u}.\partial_1 u + O(\varepsilon^2),\\
\rho\sin(\eta) &= \frac{1}{\pi}\int \dot{u}.\partial_2 u + O(\varepsilon^2),\\
A &= -\frac{1}{2\pi}\int r\dot{u}.\partial_r u + \frac{1}{2\pi}\left(\int \chi'(r)rdr\right)\int \widetilde{b}(\theta)d\theta + O(\varepsilon^2),\\
J &= -\frac{1}{2\pi}\int \dot{u}.\partial_\theta u + \frac{\rho r_c}{2\alpha}\sin(\eta-\theta_c) + O(\varepsilon^2).
\end{aligned}$$

Proof. We introduce the notation

$$h_j^{(2)} = -\frac{1}{2}\tau^{(2)}\partial_j\lambda - H_{ij}^{(2)}\partial_j\lambda, \tag{2.4.3}$$

$$h_j^{(3)} = \frac{1}{2}\partial_j\left(e^{-\lambda}\tau^{(3)}\right) - \frac{1}{2}e^{-\lambda}\tau^{(3)}\partial_j\lambda - \partial_i(e^{-\lambda}H^{(3)})_{ij} - e^{-\lambda}H_{ij}^{(3)}\partial_j\lambda. \tag{2.4.4}$$

In view of (2.3.3), (2.3.6) and (2.3.7) an easy calculation yields

$$\partial_i H_{ij}^{(1)} = f_j^{(1)} \tag{2.4.5}$$

where

$$f_j^{(1)} = e^{\lambda}\Big(- \dot{u}.\partial_j u + \frac{1}{2}\partial_j(A\Psi) - \frac{1}{2}A\Psi\partial_j\lambda + h_j^{(2)} + h_j^{(3)} + \frac{1}{2}\partial_j\tau^{(2)} - \partial_i H_{ij}^{(2)}\Big). \tag{2.4.6}$$

The three following propositions, proved respectively in Sections (2.4.1), (2.4.2) and (2.4.3) allow us to estimate the different contributions to $f_j^{(1)}$.

Proposition 2.4.2. *We have*

$$\begin{aligned}
\frac{1}{2}\partial_1\tau^{(2)} - \partial_i H_{i1}^{(2)} =& \frac{\chi'(r)}{r}b(\theta)\cos(\theta) + \frac{\chi'(r)}{r^2}\frac{r_c}{\alpha}\left(b(\theta)\sin(\theta-\theta_c)\cos(\theta)\right)', \\
\frac{1}{2}\partial_2\tau^{(2)} - \partial_i H_{i2}^{(2)} =& \frac{\chi'(r)}{r}b(\theta)\sin(\theta) + \frac{\chi'(r)}{r^2}\frac{r_c}{\alpha}\left(b(\theta)\sin(\theta-\theta_c)\sin(\theta)\right)'.
\end{aligned}$$

Proposition 2.4.3. *We have* $h_j^{(2)} \in H^0_{\delta+3}$, *with*

$$\|h_j^{(2)}\|_{H^0_{\delta+3}} \lesssim \|\lambda\|\|b\|_{W^{1,2}}.$$

Proposition 2.4.4. *We have* $h_j^{(3)} \in H^0_{\delta+3}$, *with*

$$\|h_j^{(3)}\|_{H^0_{\delta+3}} \lesssim \|B\|_{W^{1,2}}.$$

We have $e^{-\lambda}f_j^{(1)} \in H^0_{\delta+3}$:

- For $h_j^{(2)}$ and $h_j^{(3)}$ this follows from Propositions (2.4.3) and (2.4.4).
- For $\frac{1}{2}\partial_j\tau^{(2)} - \partial_i H_{j2}^{(2)}$, this is a consequence of Proposition (2.4.2). Since χ' is compactly supported, we have
$$\left\|\frac{1}{2}\partial_j\tau^{(2)} - \partial_i H_{j2}^{(2)}\right\|_{H^0_{\delta+3}} \lesssim \|b\|_{W^{1,2}(\mathbb{S}^1)}.$$
- Since $\Psi \in H^1_{\delta+2}$, we have in view of Lemma 2.2.2
$$\|A\partial_j\Psi\|_{H^0_{\delta+3}} \lesssim |A|.$$
- We have
$$\begin{aligned}
A\Psi\partial_j\lambda =& \left(-\alpha\frac{\chi(r)}{r} - r_c\cos(\theta-\theta_c)\frac{\chi(r)}{r^2} - \alpha\chi'(r)\ln(r) + r_c\cos(\theta-\theta_c)\frac{\chi'(r)}{r}\right)A\Psi\partial_j r \\
& - r_c\sin(\theta-\theta_c)\frac{\chi(r)}{r}A\Psi\partial_j\theta + A\Psi\partial_j\widetilde{\lambda},
\end{aligned}$$

and since χ' is compactly supported we have

$$\|A\Psi\partial_j\lambda\|_{H^0_{\delta+3}} \lesssim \left\| A\Psi\frac{\alpha}{1+r}\right\|_{H^0_{\delta+3}} + \left\| A\Psi\frac{r_c}{1+r^2}\right\|_{H^0_{\delta+3}} + \|A\Psi\partial_j\widetilde{\lambda}\|_{H^0_{\delta+3}}. + |A|.$$

For the terms of the form $A\Psi\frac{\alpha}{1+r}$ and $A\Psi\frac{r_c}{1+r^2}$, we use Lemma (2.2.5) which yields

$$\left\| A\Psi\frac{\alpha}{1+r}\right\|_{H^0_{\delta+3}} \lesssim |A||\alpha|, \quad \left\| A\Psi\frac{r_c}{1+r^2}\right\|_{H^0_{\delta+3}} \lesssim |A||r_c|.$$

- For the term $A\Psi\partial_j\widetilde{\lambda}$ we use Proposition (2.2.4) which yields

$$\|A\Psi\partial_j\widetilde{\lambda}\|_{H^0_{\delta+3}} \lesssim |A|\|\widetilde{\lambda}\|_{H^2_{\delta+1}}.$$

Consequently we have

$$\|e^{-\lambda}f_j^{(1)}\|_{H^0_{\delta+3}} \lesssim \|\dot{u}\nabla u\|_{H^0_{\delta+3}} + \|b\|_{W^{1,2}} + \|B\|_{W^{1,2}} + |A|.$$

We have

$$\lambda = -\alpha\chi(r)\ln(r) + \frac{r_c\cos(\theta-\theta_c)\chi(r)}{r} + \widetilde{\lambda},$$

with $\widetilde{\lambda} \in H^2_{\delta+1} \subset L^\infty$ thanks to Proposition (2.2.3). Therefore

$$|e^\lambda| \lesssim (1+r^2)^{-\frac{\alpha}{2}},$$

and Lemma (2.2.5) yields $f_j^{(1)} \in H^0_{\delta+3+\alpha}$ with

$$\|f_j^{(1)}\|_{H^0_{\delta+3+\alpha}} \lesssim \|\dot{u}\nabla u\|_{H^0_{\delta+3}} + \|b\|_{W^{1,2}} + \|B\|_{W^{1,2}} + |A|.$$

We want to solve (2.4.5) with Corollary (2.2.11). To this end, we need

$$\int f_1^{(1)} = \int f_2^{(1)} = 0. \tag{2.4.7}$$

The following proposition, proven in Section (2.4.4), allows us to carefully choose the parameters ρ, η, A in order to enforce the orthogonality condition (2.4.7).

Proposition 2.4.5. *If $\varepsilon > 0$ is small enough, there exist $\rho, \eta, A \in \mathbb{R}$ such that*

$$\int f_1^{(1)} = \int f_2^{(1)} = \int x_1 f_1^{(1)} + x_2 f_2^{(2)} = 0.$$

Moreover we have

$$\begin{aligned}
\rho\cos(\eta) &= \frac{1}{\pi}\int e^\lambda \dot{u}\partial_1 u + O(\varepsilon^2),\\
\rho\sin(\eta) &= \frac{1}{\pi}\int e^\lambda \dot{u}\partial_2 u + O(\varepsilon^2),\\
A &= -\frac{1}{2\pi}\int e^\lambda \dot{u} r\partial_r u + \frac{1}{2\pi}\left(\int \chi'(r)rdr\right)\int b(\theta)d\theta + O(\varepsilon^2).
\end{aligned}$$

We choose ρ, η, A according to Proposition (2.4.5). Since $|\alpha| \lesssim \varepsilon$, if $\varepsilon > 0$ is small enough, we have $-1 < \delta+\alpha < 0$. Since $\int f_1^{(1)} = \int f_2^{(1)} = 0$, we can apply Corollary (2.2.11). Since $\int x_1 f_1^{(1)} + x_2 f_2^{(1)} = 0$, we obtain

$$H^{(1)} = J\frac{\chi(r)}{r^2}N_\theta + \widetilde{H}^{(1)},$$

with $\widetilde{H}^{(1)} \in \mathcal{H}^1_{\delta+2+\alpha}$ such that

$$\left\|\widetilde{H}^{(1)}\right\|_{H^1_{\delta+2+\alpha}} \lesssim \left\|f_1^{(1)}\right\|_{H^0_{\delta+3+\alpha}} + \left\|f_2^{(1)}\right\|_{H^0_{\delta+3+\alpha}} \lesssim \|\dot{u}\nabla u\|_{H^0_{\delta+3}} + \|b\|_{W^{1,2}} + \|B\|_{W^{1,2}} + |A|,$$

and

$$\begin{aligned}
J =& \frac{1}{2\pi}\int x_1 f_2^{(1)} - x_2 f_1^{(1)} \\
=& \frac{1}{2\pi}\int_{\mathbb{R}^2} e^\lambda\left(-\dot{u}\partial_\theta u - A\Psi\partial_\theta\lambda + x_1(h_2^{(2)} + h_2^{(3)}) - x_2(h_1^{(2)} + h_1^{(3)})\right) + \frac{\rho r_c}{2\alpha}\sin(\eta - \theta_c) \\
&+ \frac{r_c}{\alpha}\int (e^\lambda - 1)\frac{\chi'(r)}{r}b(\theta)\sin(\theta - \theta_c) \\
=& -\frac{1}{2\pi}\int_{\mathbb{R}^2} e^\lambda \dot{u}\partial_\theta u + \rho\frac{r_c}{2\alpha}\sin(\theta - \theta_c) + O(\varepsilon^2)
\end{aligned} \tag{2.4.8}$$

where we have used the definition (2.4.6) of $f_j^{(1)}$, $x_1\partial_2 - x_2\partial_1 = \partial_\theta$, Proposition 2.4.2 and the following calculations

$$\frac{1}{2}\int e^\lambda A\partial_\theta\Psi = -\frac{1}{2}\int e^\lambda A\Psi\partial_\theta\lambda,$$

$$\begin{aligned}
&\int e^\lambda\left(x_1\frac{\chi'(r)}{r}b(\theta)\sin(\theta) - x_2\frac{\chi'(r)}{r}b(\theta)\cos(\theta)\right) \\
=& \int e^\lambda\chi'(r)b(\theta)(\cos(\theta)\sin(\theta) - \cos(\theta)\sin(\theta))rdrd\theta \\
=& 0,
\end{aligned}$$

$$\begin{aligned}
&\int e^\lambda\frac{\chi'(r)}{r^2}\frac{r_c}{\alpha}\left(x_1\left(b(\theta)\sin(\theta - \theta_c)\sin(\theta)\right)' - x_2\left(b(\theta)\sin(\theta - \theta_c)\cos(\theta)\right)'\right) \\
=& -\frac{r_c}{\alpha}\int e^\lambda\frac{\chi'(r)}{r}b(\theta)\sin(\theta - \theta_c)(-\sin^2(\theta) - \cos^2(\theta)) \\
&- \frac{r_c}{\alpha}\int \partial_\theta\lambda e^\lambda\frac{\chi'(r)}{r}b(\theta)\sin(\theta - \theta_c)(\cos(\theta)\sin(\theta) - \sin(\theta)\cos(\theta)) \\
=& \frac{r_c}{\alpha}\left(\int\chi'(r)dr\right)\left(\int b(\theta)\sin(\theta - \theta_c)d\theta\right) + \frac{r_c}{\alpha}\int(e^\lambda - 1)\frac{\chi'(r)}{r}b(\theta)\sin(\theta - \theta_c) \\
=& \pi\frac{\rho r_c}{\alpha}\sin(\eta - \theta_c) + \frac{r_c}{\alpha}\int(e^\lambda - 1)\frac{\chi'(r)}{r}b(\theta)\sin(\theta - \theta_c),
\end{aligned}$$

where we have used in the last equality the definition of b (2.3.10) and the orthogonality condition (2.3.1) for $\widetilde{b}$. It remains to estimate $e^{-\lambda}\widetilde{H}^{(1)}$ in $\mathcal{H}^1_{\delta+2}$. First, we note that since

$$-\lambda - \alpha\chi(r)\ln(r)$$

is bounded, thanks to Lemma (2.2.5) and the fact that $\widetilde{H}^{(1)} \in \mathcal{H}^1_{\delta+2+\alpha}$ we have $e^{-\lambda}\widetilde{H}^{(1)} \in \mathcal{H}^0_{\delta+2}$. We now calculate $\nabla(e^{-\lambda}\widetilde{H}^{(1)})$. The contributions are

- the term $e^{-\lambda}\nabla\widetilde{H}^{(1)}$: since $\nabla\widetilde{H}^{(1)} \in \mathcal{H}^0_{\delta+\alpha+3}$, we have $e^{-\lambda}\nabla\widetilde{H}^{(1)} \in \mathcal{H}^0_{\delta+3}$ thanks to Lemma (2.2.5),
- the term $\frac{\alpha\chi(r)}{r}e^{-\lambda}\widetilde{H}^{(1)}$: it also belongs to $\mathcal{H}^0_{\delta+3}$ thanks to Lemma (2.2.5).
- The term $e^{-\lambda}\widetilde{H}^{(1)}\nabla\widetilde{\lambda}$: thanks to Proposition (2.2.4), $\widetilde{H}^{(1)}\nabla\widetilde{\lambda}$ belong to $\mathcal{H}^0_{\delta+3+\alpha}$, and therefore, thanks to Lemma (2.2.5), we have $e^{-\lambda}\widetilde{H}^{(1)}\nabla\widetilde{\lambda} \in \mathcal{H}^0_{\delta+3}$.

Consequently, we have $\nabla(e^{-\lambda}\widetilde{H}^{(1)}) \in \mathcal{H}^0_{\delta+3}$ and therefore $e^{-\lambda}\widetilde{H}^{(1)} \in \mathcal{H}^1_{\delta+2}$ with

$$\|e^{-\lambda}\widetilde{H}^{(1)}\|_{\mathcal{H}^1_{\delta+2}} \lesssim \|\dot{u}\nabla u\|_{H^0_{\delta+3}} + \|b\|_{W^{1,2}} + \|B\|_{W^{1,2}} + |A| \lesssim \varepsilon.$$

This concludes the proof of Proposition (2.4.1). □

2.4.1 Proof of Proposition (2.4.2)

We calculate

$$\begin{aligned}
\partial_i\left(b(\theta)\frac{-\chi(r)}{2r}M_\theta\right)_{i1} =& -\frac{b(\theta)}{2}\left(\frac{\chi'(r)}{r}-\frac{\chi(r)}{r^2}\right)(\cos(\theta)\cos(2\theta)+\sin(\theta)\sin(2\theta))\\
&-\frac{b(\theta)\chi(r)}{r^2}(\sin(\theta)\sin(2\theta)+\cos(\theta)\cos(2\theta))\\
&-\frac{b'(\theta)\chi(r)}{2r^2}(-\sin(\theta)\cos(2\theta)+\cos(\theta)\sin(2\theta))\\
=& -\frac{b(\theta)\chi(r)}{2r^2}\cos(\theta)-\frac{b'(\theta)\chi(r)}{2r^2}\sin(\theta)-\frac{b(\theta)\chi'(r)}{2r}\cos(\theta),\\
\frac{1}{2}\partial_1\left(b(\theta)\frac{\chi(r)}{r}\right) =& \frac{1}{2}b(\theta)\left(\frac{\chi'(r)}{r}-\frac{\chi(r)}{r^2}\right)\cos(\theta)-\frac{1}{2}b'(\theta)\frac{\chi(r)}{r^2}\sin(\theta).
\end{aligned}$$

Therefore

$$\frac{1}{2}\partial_1\left(b(\theta)\frac{\chi(r)}{r}\right)-\partial_i\left(b(\theta)\frac{-\chi(r)}{2r}M_\theta\right)_{i1} = \frac{b(\theta)\chi'(r)}{r}\cos(\theta).$$

For $j=2$ we obtain

$$\begin{aligned}
\partial_i\left(b(\theta)\frac{-\chi(r)}{2r}M_\theta\right)_{i2} =& -\frac{b(\theta)}{2}\left(\frac{\chi'(r)}{r}-\frac{\chi(r)}{r^2}\right)(\cos(\theta)\sin(2\theta)-\sin(\theta)\cos(2\theta))\\
&-\frac{b(\theta)\chi(r)}{r^2}(-\sin(\theta)\cos(2\theta)+\cos(\theta)\sin(2\theta))\\
&-\frac{b'(\theta)\chi(r)}{2r^2}(-\sin(\theta)\sin(2\theta)-\cos(\theta)\cos(2\theta))\\
=& -\frac{b(\theta)\chi(r)}{2r^2}\sin(\theta)+\frac{b'(\theta)\chi(r)}{2r^2}\cos(\theta)-\frac{b(\theta)\chi'(r)}{2r}\sin(\theta),\\
\frac{1}{2}\partial_2\left(b(\theta)\frac{\chi(r)}{r}\right) =& \frac{1}{2}b(\theta)\left(\frac{\chi'(r)}{r}-\frac{\chi(r)}{r^2}\right)\sin(\theta)+\frac{1}{2}b'(\theta)\frac{\chi(r)}{r^2}\cos(\theta).
\end{aligned}$$

Therefore

$$\frac{1}{2}\partial_2\left(b(\theta)\frac{\chi(r)}{r}\right)-\partial_i\left(b(\theta)\frac{-\chi(r)}{2r}M_\theta\right)_{i2} = \frac{b(\theta)\chi'(r)}{r}\sin(\theta).$$

We now calculate the other contributions. We note that $\frac{1}{r^2}M_\theta$ and $\frac{1}{r^2}N_\theta$ satisfy

$$\partial_i\left(\frac{1}{r^2}M_\theta\right)_{ij} = \partial_i\left(\frac{1}{r^2}N_\theta\right)_{ij} = 0, \quad for\ r>0. \tag{2.4.9}$$

This yields

$$
\begin{aligned}
&\partial_i \left(-\frac{r_c}{2\alpha}(b(\theta)\sin(\theta-\theta_c))'\frac{\chi(r)}{r^2}M_\theta - \frac{r_c}{\alpha}b(\theta)\sin(\theta-\theta_c)\frac{\chi(r)}{r^2}N_\theta \right)_{i1} \\
=& -\frac{r_c}{2\alpha}(b(\theta)\sin(\theta-\theta_c))'(\cos(\theta)\cos(2\theta)+\sin(\theta)\sin(2\theta))\frac{\chi'(r)}{r^2} \\
&-\frac{r_c}{\alpha}b(\theta)\sin(\theta-\theta_c)(-\cos(\theta)\sin(2\theta)+\sin(\theta)\cos(2\theta))\frac{\chi'(r)}{r^2} \\
&-\frac{r_c}{2\alpha}(b(\theta)\sin(\theta-\theta_c))''(-\sin(\theta)\cos(2\theta)+\cos(\theta)\sin(2\theta))\frac{\chi(r)}{r^3} \\
&-\frac{r_c}{\alpha}(b(\theta)\sin(\theta-\theta_c))'(\sin(\theta)\sin(2\theta)+\cos(\theta)\cos(2\theta))\frac{\chi(r)}{r^3} \\
=&\frac{r_c}{\alpha}\left(b(\theta)\sin(\theta-\theta_c)\sin(\theta) - \frac{1}{2}(b(\theta)\sin(\theta-\theta_c))'\cos(\theta) \right)\frac{\chi'(r)}{r^2} \\
&+\frac{r_c}{\alpha}\left(-(b(\theta)\sin(\theta-\theta_c))'\cos(\theta) - \frac{1}{2}(b(\theta)\sin(\theta-\theta_c))''\sin(\theta) \right)\frac{\chi(r)}{r^3}.
\end{aligned}
$$

We now calculate the term involving τ.

$$
\begin{aligned}
&\frac{1}{2}\partial_1 \left(\frac{r_c}{\alpha}(b(\theta)\sin(\theta-\theta_c))'\frac{\chi(r)}{r^2} \right) \\
=&\frac{r_c}{2\alpha}\left(\frac{\chi'(r)}{r^2} - 2\frac{\chi(r)}{r^3} \right)\cos(\theta)(b(\theta)\sin(\theta-\theta_c))' + \frac{r_c}{2\alpha}(-\sin(\theta))(b(\theta)\sin(\theta-\theta_c))''\frac{\chi(r)}{r^3} \\
=&\frac{r_c}{2\alpha}\cos(\theta)(b(\theta)\sin(\theta-\theta_c))'\frac{\chi'(r)}{r^2} \\
&+\frac{r_c}{\alpha}\left(-(b(\theta)\sin(\theta-\theta_c))'\cos(\theta) - \frac{1}{2}(b(\theta)\sin(\theta-\theta_c))''\sin(\theta) \right)\frac{\chi(r)}{r^3}.
\end{aligned}
$$

Therefore we have

$$
\begin{aligned}
&\frac{1}{2}\partial_1 \left(\frac{r_c}{\alpha}(b(\theta)\sin(\theta-\theta_c))'\frac{\chi(r)}{r^2} \right) \\
&-\partial_i \left(-\frac{r_c}{2\alpha}(b(\theta)\sin(\theta-\theta_c))'\frac{\chi(r)}{r^2}M_\theta - \frac{r_c}{\alpha}b(\theta)\sin(\theta-\theta_c)\frac{\chi(r)}{r^2}N_\theta \right)_{i1} \\
=&\frac{r_c}{\alpha}\left(\cos(\theta)(b(\theta)\sin(\theta-\theta_c))' - \sin(\theta)b(\theta)\sin(\theta-\theta_c) \right)\frac{\chi'(r)}{r^2} \\
=&\frac{r_c}{\alpha}\left(\cos(\theta)b(\theta)\sin(\theta-\theta_c) \right)'\frac{\chi'(r)}{r^2}.
\end{aligned}
$$

For $j = 2$ we obtain

$$\begin{aligned}
&\partial_i \left(-\frac{r_c}{2\alpha}(b(\theta)\sin(\theta-\theta_c))'\frac{\chi(r)}{r^2}M_\theta - \frac{r_c}{\alpha}b(\theta)\sin(\theta-\theta_c)\frac{\chi(r)}{r^2}N_\theta \right)_{i2} \\
=& -\frac{r_c}{2\alpha}(b(\theta)\sin(\theta-\theta_c))'(\cos(\theta)\sin(2\theta)-\sin(\theta)\cos(2\theta))\frac{\chi'(r)}{r^2} \\
&-\frac{r_c}{\alpha}b(\theta)\sin(\theta-\theta_c)(\cos(\theta)\cos(2\theta)+\sin(\theta)\sin(2\theta))\frac{\chi'(r)}{r^2} \\
&-\frac{r_c}{2\alpha}(b(\theta)\sin(\theta-\theta_c))''(-\sin(\theta)\sin(2\theta)-\cos(\theta)\cos(2\theta))\frac{\chi(r)}{r^3} \\
&-\frac{r_c}{\alpha}(b(\theta)\sin(\theta-\theta_c))'(-\sin(\theta)\cos(2\theta)+\cos(\theta)\sin(2\theta))\frac{\chi(r)}{r^3} \\
=&\frac{r_c}{\alpha}\left(-b(\theta)\sin(\theta-\theta_c)\cos(\theta)-\frac{1}{2}(b(\theta)\sin(\theta-\theta_c))'\sin(\theta)\right)\frac{\chi'(r)}{r^2} \\
&+\frac{r_c}{\alpha}\left(-(b(\theta)\sin(\theta-\theta_c))'\sin(\theta)+\frac{1}{2}(b(\theta)\sin(\theta-\theta_c))''\cos(\theta)\right)\frac{\chi(r)}{r^3}.
\end{aligned}$$

We now calculate the term involving τ.

$$\begin{aligned}
&\frac{1}{2}\partial_2\left(\frac{r_c}{\alpha}(b(\theta)\sin(\theta-\theta_c))'\frac{\chi(r)}{r^2}\right) \\
=&\frac{r_c}{2\alpha}\left(\frac{\chi'(r)}{r^2}-2\frac{\chi(r)}{r^3}\right)\sin(\theta)(b(\theta)\sin(\theta-\theta_c))' + \frac{r_c}{2\alpha}\cos(\theta)(b(\theta)\sin(\theta-\theta_c))''\frac{\chi(r)}{r^3} \\
=&\frac{r_c}{2\alpha}\sin(\theta)(b(\theta)\sin(\theta-\theta_c))'\frac{\chi'(r)}{r^2} \\
&+\frac{r_c}{\alpha}\left(-(b(\theta)\sin(\theta-\theta_c))'\sin(\theta)+\frac{1}{2}(b(\theta)\sin(\theta-\theta_c))''\cos(\theta)\right)\frac{\chi(r)}{r^3}.
\end{aligned}$$

Therefore we have

$$\begin{aligned}
&\frac{1}{2}\partial_2\left(\frac{r_c}{\alpha}(b(\theta)\sin(\theta-\theta_c))'\frac{\chi(r)}{r^2}\right) \\
&-\partial_i\left(-\frac{r_c}{2\alpha}(b(\theta)\sin(\theta-\theta_c))'\frac{\chi(r)}{r^2}M_\theta - \frac{r_c}{\alpha}b(\theta)\sin(\theta-\theta_c)\frac{\chi(r)}{r^2}N_\theta\right)_{i2} \\
=&\frac{r_c}{\alpha}\left(\sin(\theta)(b(\theta)\sin(\theta-\theta_c))'+\cos(\theta)b(\theta)\sin(\theta-\theta_c)\right)\frac{\chi'(r)}{r^2} \\
=&\frac{r_c}{\alpha}\left(\sin(\theta)b(\theta)\sin(\theta-\theta_c)\right)'\frac{\chi'(r)}{r^2}.
\end{aligned}$$

In view of (2.3.4) and (2.3.8), this concludes the proof of Proposition (2.4.2).

2.4.2 Proof of Proposition (2.4.3)

Since $|\nabla\widetilde{\lambda}| \in H^1_{\delta+2}$, Lemma (2.2.5) implies that the terms of the form $\frac{|b|}{1+r}|\nabla\widetilde{\lambda}|$ belong to $H^0_{\delta+3}$ and satisfy

$$\left\|\frac{|b|}{1+r}|\nabla\widetilde{\lambda}|\right\|_{H^0_{\delta+3}} \lesssim \|b\|_{L^\infty(\mathbb{S}^1)}\|\nabla\widetilde{\lambda}\|_{H^0_{\delta+2}}$$

and consequently, with the Sobolev injection $W^{1,2}(\mathbb{S}^1) \subset L^\infty(\mathbb{S}^1)$,

$$\left\|\frac{|b|}{1+r}|\nabla\widetilde{\lambda}|\right\|_{H^0_{\delta+3}} \lesssim \|b\|_{W^{1,2}(\mathbb{S}^1)}\|\widetilde{\lambda}\|_{H^1_{\delta+1}}. \tag{2.4.10}$$

Moreover, thanks to Lemma (2.2.6), the terms of the form $\frac{r_c}{\alpha}\frac{|b|+|b'|}{(1+r)^2}|\nabla\widetilde{\lambda}|$ belong to $H^0_{\delta+3}$ and satisfy

$$\left\|\frac{r_c}{\alpha}\frac{|b|+|b'|}{(1+r)^2}|\nabla\widetilde{\lambda}|\right\|_{H^0_{\delta+3}} \lesssim \|b\|_{W^{1,2}(\mathbb{S}^1)}\|\widetilde{\lambda}\|_{H^2_{\delta+1}}, \tag{2.4.11}$$

where we have used, that thanks to (2.4.2), $\frac{r_c}{\alpha}\lesssim 1$. The terms of the form $\frac{r_c}{\alpha}\frac{|b|+|b'|}{(1+r)^2}\frac{r_c}{(1+r)^2}$ are also in $H^0_{\delta+3}$ and satisfy

$$\left\|\frac{r_c}{\alpha}\frac{|b|+|b'|}{(1+r)^2}\frac{r_c}{(1+r)^2}\right\|_{H^0_{\delta+3}} \lesssim \|b\|_{W^{1,2}(\mathbb{S}^1)}|r_c|. \tag{2.4.12}$$

Finally, since χ' is compactly supported, we also have

$$\left\|\frac{|b|}{1+r}\alpha\chi'(r)\ln(r)\right\|_{H^0_{\delta+3}} \lesssim \|b\|_{W^{1,2}(\mathbb{S}^1)}|\alpha|, \quad \left\|\frac{|b|}{1+r}\frac{r_c\chi'(r)}{(1+r)}\right\|_{H^0_{\delta+3}} \lesssim \|b\|_{W^{1,2}(\mathbb{S}^1)}|r_c|, \quad ... \tag{2.4.13}$$

Consequently, the terms which remain to calculate are the ones decaying like $\frac{1}{r^2}$ and $\frac{1}{r^3}$. We obtain

$$\begin{aligned}
H^{(2)}_{i1}\partial_i\lambda = & -\frac{\chi(r)}{2r}b(\theta)\Bigg(-\left(\frac{\alpha}{r}+\frac{r_c\cos(\theta-\theta_c)}{r^2}\right)(\cos(2\theta)\cos(\theta)+\sin(2\theta)\sin(\theta)) \\
& -\frac{r_c\sin(\theta-\theta_c)}{r^2}(-\sin(\theta)\cos(2\theta)+\cos(\theta)\sin(2\theta))\Bigg) \\
& -\frac{r_c\chi(r)}{2\alpha r^2}(\cos(\theta-\theta_c)b(\theta)+\sin(\theta-\theta_c)b'(\theta))\frac{-\alpha}{r}(\cos(2\theta)\cos(\theta)+\sin(2\theta)\sin(\theta)) \\
& -\frac{r_c\chi(r)}{\alpha r^2}\sin(\theta-\theta_c)b(\theta)\frac{-\alpha}{r}(-\cos(\theta)\sin(2\theta)+\sin(\theta)\cos(2\theta))+h_1 \\
= & \frac{\alpha b(\theta)\chi(r)}{2r^2}\cos(\theta)+\frac{r_cb(\theta)\chi(r)}{r^3}\left(\cos(\theta)\cos(\theta-\theta_c)-\frac{1}{2}\sin(\theta)\sin(\theta-\theta_c)\right) \\
& +\frac{r_cb'(\theta)\chi(r)}{2r^3}\cos(\theta)\sin(\theta-\theta_c)+h_1,
\end{aligned}$$

where, thanks to (2.4.10), (2.4.11) and (2.4.12), $h_1\in H^0_{\delta+3}$ satisfies

$$\|h_1\|_{H^0_{\delta+3}} \lesssim \|\lambda\|\|b\|_{W^{1,2}}.$$

We calculate

$$\begin{aligned}
-\frac{1}{2}\tau^{(2)}\partial_1\lambda = & -\frac{b(\theta)\chi(r)}{2r}\left(-\frac{\alpha}{r}\cos(\theta)-\frac{r_c\cos(\theta-\theta_c)}{r^2}\cos(\theta)+\frac{r_c\sin(\theta-\theta_c)}{r^2}\sin(\theta)\right) \\
& -\frac{r_c}{2\alpha}(b(\theta)\cos(\theta-\theta_c)+b'(\theta)\sin(\theta-\theta_c))\frac{\chi(r)}{r^2}\left(\frac{-\alpha}{r}\cos(\theta)\right)+h_2 \\
= & \frac{\alpha b(\theta)\chi(r)}{2r^2}\cos(\theta)+\frac{r_cb(\theta)\chi(r)}{r^3}\left(\cos(\theta)\cos(\theta-\theta_c)-\frac{1}{2}\sin(\theta)\sin(\theta-\theta_c)\right) \\
& +\frac{r_cb'(\theta)\chi(r)}{2r^3}\cos(\theta)\sin(\theta-\theta_c)+h_2,
\end{aligned}$$

where thanks to (2.4.10), (2.4.11) and (2.4.12), $h_2\in H^0_{\delta+3}$ satisfies

$$\|h_2\|_{h^0_{\delta+3}} \lesssim \|\lambda\|\|b\|_{W^{1,2}}.$$

Therefore

$$-\frac{1}{2}\tau^{(2)}\partial_1\lambda - H^{(2)}_{i1}\partial_i\lambda = h_2-h_1\in H^0_{\delta+3}.$$

For $j = 2$ we obtain

$$\begin{aligned}
H^{(2)}_{i2}\partial_i\lambda = &-\frac{\chi(r)}{2r}b(\theta)\Bigg(-\left(\frac{\alpha}{r}+\frac{r_c\cos(\theta-\theta_c)}{r^2}\right)(-\cos(2\theta)\sin(\theta)+\sin(2\theta)\cos(\theta))\\
&-\frac{r_c\sin(\theta-\theta_c)}{r^2}(-\cos(\theta)\cos(2\theta)-\sin(\theta)\sin(2\theta))\Bigg)\\
&-\frac{r_c\chi(r)}{2\alpha r^2}(\cos(\theta-\theta_c)b(\theta)+\sin(\theta-\theta_c)b'(\theta)\frac{-\alpha}{r}(-\cos(2\theta)\sin(\theta)+\sin(2\theta)\cos(\theta))\\
&-\frac{r_c\chi(r)}{\alpha r^2}\sin(\theta-\theta_c)b(\theta)\frac{-\alpha}{r}(\sin(\theta)\sin(2\theta)+\cos(\theta)\cos(2\theta))+h_3\\
=&\frac{\alpha b(\theta)\chi(r)}{2r^2}\sin(\theta)+\frac{r_cb(\theta)\chi(r)}{r^3}\left(\sin(\theta)\cos(\theta-\theta_c)+\frac{1}{2}\cos(\theta)\sin(\theta-\theta_c)\right)\\
&+\frac{r_cb'(\theta)\chi(r)}{2r^3}\sin(\theta)\sin(\theta-\theta_c)+h_3,
\end{aligned}$$

where thanks to (2.4.10), (2.4.11) and (2.4.12), $h_3 \in H^0_{\delta+3}$ satisfies

$$\|h_3\|_{H^0_{\delta+3}} \lesssim \|\lambda\|\|b\|_{W^{1,2}}.$$

We calculate

$$\begin{aligned}
-\frac{1}{2}\tau^{(2)}\partial_2\lambda = &-\frac{b(\theta)\chi(r)}{2r}\left(-\frac{\alpha}{r}\sin(\theta)-\frac{r_c\cos(\theta-\theta_c)}{r^2}\sin(\theta)-\frac{r_c\sin(\theta-\theta_c)}{r^2}\cos(\theta)\right)\\
&-\frac{r_c}{2\alpha}(b(\theta)\cos(\theta-\theta_c)+b'(\theta)\sin(\theta-\theta_c))\frac{\chi(r)}{r^2}\left(\frac{-\alpha}{r}\sin(\theta)\right)+h_4\\
=&\frac{\alpha b(\theta)\chi(r)}{2r^2}\sin(\theta)+\frac{r_cb(\theta)\chi(r)}{r^3}\left(\sin(\theta)\cos(\theta-\theta_c)+\frac{1}{2}\cos(\theta)\sin(\theta-\theta_c)\right)\\
&+\frac{r_cb'(\theta)\chi(r)}{2r^3}\sin(\theta)\sin(\theta-\theta_c)+h_4,
\end{aligned}$$

where thanks to (2.4.10), (2.4.11) and (2.4.12), $h_4 \in H^0_{\delta+3}$ satisfies

$$\|h_4\|_{H^0_{\delta+3}} \lesssim \|\lambda\|\|b\|_{W^{1,\infty}}.$$

Therefore

$$-\frac{1}{2}\tau^{(2)}\partial_2\lambda - H^{(2)}_{i2}\partial_i\lambda = h_4 - h_3 \in H^0_{\delta+3}.$$

This concludes the proof of Proposition (2.4.3).

2.4.3 Proof of Proposition (2.4.4)

We first note

$$\partial_i(e^{-\lambda}H^{(3)})_{ij} + e^{-\lambda}H^{(3)}_{ij}\partial_j\lambda = e^{-\lambda}\partial_iH^{(3)}_{ij},$$

$$\frac{1}{2}\partial_j\left(e^{-\lambda}\tau^{(3)}\right) - \frac{1}{2}e^{-\lambda}\tau^{(3)}\partial_j\lambda = \frac{1}{2}e^{-\lambda}\partial_j\tau^{(3)} - e^{-\lambda}\tau^{(3)}\partial_j\lambda.$$

Since $\widetilde{\lambda} \in H^2_{\delta+1}$, Proposition (2.2.3) implies that $\widetilde{\lambda}$ is bounded and consequently

$$|e^{-\lambda}| \lesssim (1+r^2)^{\frac{\alpha}{2}}.$$

Therefore Lemma (2.2.6) imply that the terms of the form $e^{-\lambda}\frac{|B|+|B'|}{(1+r^2)}\nabla\widetilde{\lambda}$ belong to $H^0_{\delta+4-\alpha}$, with

$$\left\|e^{-\lambda}\frac{|B|+|B'|}{(1+r^2)}\nabla\widetilde{\lambda}\right\|_{H^0_{\delta+4-\alpha}} \lesssim \|B\|_{W^{1,2}(\mathbb{S}^1)}\|\widetilde{\lambda}\|_{H^2_{\delta+1}}.$$

Since α is of size ε, for ε small enough we have $\alpha<1$ and

$$\left\|e^{-\lambda}\frac{|B|+|B'|}{(1+r^2)}\nabla\widetilde{\lambda}\right\|_{H^0_{\delta+3}} \lesssim \|B\|_{W^{1,2}(\mathbb{S}^1)}\|\widetilde{\lambda}\|_{H^2_{\delta+1}}. \tag{2.4.14}$$

The terms of the form $e^{-\lambda}\frac{|B|+|B'|}{1+r^2}\frac{r_c}{1+r^2}$ satisfy

$$\left|e^{-\lambda}\frac{|B|+|B'|}{1+r^2}\frac{r_c}{1+r^2}\right| \lesssim \frac{r_c(|B|+|B'|)}{(1+r^2)^{2-\frac{\alpha}{2}}},$$

so, for $\varepsilon>0$ small enough so that $\delta+\alpha<0$ they belong to $H^0_{\delta+3}$ and satisfy

$$\left\|e^{-\lambda}\frac{|B|+|B'|}{1+r^2}\frac{r_c}{1+r^2}\right\|_{H^0_{\delta+3}} \lesssim \|B\|_{W^{1,2}(\mathbb{S}^1)}|r_c|. \tag{2.4.15}$$

Since χ' is smooth and compactly supported, the term of the form $e^{-\lambda}\frac{|B|+|B'|}{1+r^2}\chi'(r)$ belong to $H^0_{\delta+3}$ and satisfy

$$\left\|e^{-\lambda}\frac{|B|+|B'|}{1+r^2}\chi'(r)\right\|_{H^0_{\delta+3}} \lesssim \|B\|_{W^{1,2}(\mathbb{S}^1)}. \tag{2.4.16}$$

Consequently the terms which remain to calculate are the one which decay like $\frac{r^\alpha}{r^3}$. We calculate

$$\begin{aligned}
e^{-\lambda}\partial_i H^{(3)}_{i1} =& -e^{-\lambda}(1-\alpha)B'(\theta)(\sin(\theta)\sin(2\theta)+\cos(\theta)\cos(2\theta))\frac{\chi(r)}{r^3}\\
&-e^{-\lambda}\frac{B''(\theta)}{2}(-\sin(\theta)\cos(2\theta)+\cos(\theta)\sin(2\theta))\frac{\chi(r)}{r^3}+g_1\\
=&e^{-\lambda}\left(-(1-\alpha)B'(\theta)\cos(\theta)-\frac{1}{2}B''(\theta)\sin(\theta)\right)\frac{\chi(r)}{r^3}+g_1
\end{aligned}$$

where we have used (2.4.9) and where, thanks to the estimate (2.4.16), $g_1\in H^0_{\delta+3}$ satisfies

$$\|g_1\|_{H^0_{\delta+3}} \lesssim \|B\|_{W^{1,2}}.$$

We now calculate

$$\begin{aligned}
&\frac{1}{2}e^{-\lambda}\partial_1\tau^{(3)}-\tau^{(3)}e^{-\lambda}\partial_1\lambda\\
&=-2\frac{1}{2}e^{-\lambda}B'(\theta)\frac{\chi(r)}{r^3}\cos(\theta)-\frac{1}{2}B''(\theta)\frac{e^{-\lambda}\chi(r)}{r^3}\sin(\theta)-B'(\theta)e^{-\lambda}\frac{\chi(r)}{r^2}\frac{-\alpha}{r}\cos(\theta)+g_2\\
&=(\alpha-1)e^{-\lambda}B'(\theta)\frac{\chi(r)}{r^3}\cos(\theta)-\frac{1}{2}e^{-\lambda}B''(\theta)\frac{\chi(r)}{r^3}\sin(\theta)+g_2
\end{aligned}$$

where thanks to the estimates (2.4.14), (2.4.15) and (2.4.16), $g_2\in H^0_{\delta+3}$ satisfies

$$\|g_2\|_{H^0_{\delta+3}} \lesssim \|B\|_{W^{1,2}}.$$

Therefore

$$\frac{1}{2}\partial_1\tau^{(3)}-\frac{1}{2}\tau^{(3)}\partial_1\lambda-\partial_i(e^{-\lambda}H^{(3)})_{1j}-e^{-\lambda}H^{(3)}_{1j}\partial_j\lambda = g_2-g_1\in H^0_{\delta+3}.$$

For $j = 2$ we have

$$\begin{aligned}
e^{-\lambda}\partial_i H^{(3)}_{i2} =& - e^{-\lambda}(1-\alpha)B'(\theta)(-\sin(\theta)\cos(2\theta)+\cos(\theta)\sin(2\theta))\frac{\chi(r)}{r^3}\\
&- e^{-\lambda}\frac{B''(\theta)}{2}(-\sin(\theta)\sin(2\theta)-\cos(\theta)\cos(2\theta))\frac{\chi(r)}{r^3}+g_3\\
=& e^{-\lambda}\left(-(1-\alpha)B'(\theta)\sin(\theta)+\frac{1}{2}B''(\theta)\cos(\theta)\right)\frac{\chi(r)}{r^3}+g_3
\end{aligned}$$

where thanks to the estimate (2.4.16), $g_3 \in H^0_{\delta+3}$ satisfies

$$\|g_3\|_{H^0_{\delta+3}} \lesssim \|B\|_{W^{1,2}}.$$

$$\begin{aligned}
&\frac{1}{2}e^{-\lambda}\partial_2\tau^{(3)} - e^{-\lambda}\tau^{(3)}\partial_2\lambda\\
&= -2e^{-\lambda}\frac{1}{2}B'(\theta)\frac{\chi(r)}{r^3}\sin(\theta)+\frac{1}{2}B''(\theta)\frac{e^{-\lambda}\chi(r)}{r^3}\cos(\theta)-B'(\theta)e^{-\lambda}\frac{\chi(r)}{r^2}\frac{-\alpha}{r}\sin(\theta)+g_4\\
&= (\alpha-1)B'(\theta)\frac{\chi(r)e^{-\lambda}}{r^3}\sin(\theta)+\frac{1}{2}B''(\theta)\frac{\chi(r)e^{-\lambda}}{r^3}\cos(\theta)+g_4
\end{aligned}$$

where thanks to the estimates (2.4.14), (2.4.15) and (2.4.16), $g_4 \in H^0_{\delta+3}$ satisfies

$$\|g_4\|_{H^0_{\delta+3}} \lesssim \|B\|_{W^{1,2}}.$$

Therefore

$$\frac{1}{2}\partial_2\tau^{(3)} - \frac{1}{2}\tau^{(3)}\partial_2\lambda - \partial_i(e^{-\lambda}H^{(3)})_{i2} - e^{-\lambda}H^{(3)}_{i2}\partial_i\lambda = g_4 - g_3 \in H^0_{\delta+3}.$$

This conclude the proof of Proposition (2.4.4).

2.4.4 Proof of Proposition (2.4.5)

Recall that $f^{(1)}_j$ has been defined in (2.4.6). We calculate

$$\begin{aligned}
\int_{\mathbb{R}^2} f^{(1)}_1 =& \int_{\mathbb{R}^2} e^{\lambda}\left(-\dot{u}.\partial_1 u - A\Psi\partial_1\lambda + h^{(2)}_1 + h^{(3)}_1\right)\\
&+\int (e^{\lambda}-1)\frac{\chi'(r)}{r}b(\theta)\cos(\theta)dx - \frac{r_c}{\alpha}\int e^{\lambda}\frac{\chi'(r)}{r^2}b(\theta)\sin(\theta-\theta_c)\cos(\theta)\partial_\theta\lambda\\
&+\pi\rho\cos(\eta),
\end{aligned} \tag{2.4.17}$$

where we have used Proposition 2.4.2 and the calculations

$$\frac{1}{2}\int e^{\lambda}A\partial_1\Psi = -\frac{1}{2}\int e^{\lambda}A\Psi\partial_1\lambda,$$

$$\begin{aligned}
\int e^{\lambda}\frac{\chi'(r)}{r}b(\theta)\cos(\theta) &= \int (e^{\lambda}-1)\frac{\chi'(r)}{r}b(\theta)\cos(\theta) + \left(\int \chi'(r)dr\right)\left(\int b(\theta)\cos(\theta)d\theta\right)\\
&= \int (e^{\lambda}-1)\frac{\chi'(r)}{r}b(\theta)\cos(\theta)+\pi\rho\cos(\eta),
\end{aligned}$$

where we have used the definition of b (2.3.10) and the orthogonality condition 2.3.1,

$$\frac{r_c}{\alpha}\int e^{\lambda}\frac{\chi'(r)}{r^2}(b(\theta)\sin(\theta-\theta_c)\cos(\theta))' = -\frac{r_c}{\alpha}\int e^{\lambda}\frac{\chi'(r)}{r^2}b(\theta)\sin(\theta-\theta_c)\cos(\theta)\partial_\theta\lambda.$$

Similarly, we have

$$\begin{aligned}\int_{\mathbb{R}^2} f_2^{(1)} &= \int_{\mathbb{R}^2} e^{\lambda}\left(-\dot{u}\partial_2 u - A\Psi\partial_2\lambda + h_2^{(2)} + h_2^{(3)}\right)\\ &+ \int (e^{\lambda}-1)\frac{\chi'(r)}{r}b(\theta)\sin(\theta)dx - \frac{r_c}{\alpha}\int e^{\lambda}\frac{\chi'(r)}{r^2}b(\theta)\sin(\theta-\theta_c)\sin(\theta)\partial_\theta\lambda\\ &+ \pi\rho\sin(\eta).\end{aligned} \tag{2.4.18}$$

We calculate also

$$\begin{aligned}\int_{\mathbb{R}^2} x_1 f^{(1)} + x_2 f^{(2)} = &\int_{\mathbb{R}^2} e^{\lambda}\left(\dot{u}(r\partial_r u) - A\Psi r\partial_r\lambda + x_1(h_1^{(2)} + h_1^{(3)}) + x_2(h_2^{(2)} + h_2^{(3)})\right)\\ &+ \int (e^{\lambda}-1)\chi'(r)b(\theta) - \frac{r_c}{\alpha}\int e^{\lambda}\partial_\theta\lambda\frac{\chi'(r)}{r}b(\theta)\sin(\theta-\theta_c)\\ &+ \left(\int \chi'(r)rdr\right)\int b(\theta)d\theta - A\int e^{\lambda}\Psi\end{aligned} \tag{2.4.19}$$

where we used $x_1\partial_1 + x_2\partial_2 = r\partial_r$ and the following calculations

$$\frac{1}{2}\int e^{\lambda}(x_1\partial_1 A\Psi + x_2\partial_2 A\Psi) = -\frac{1}{2}\int e^{\lambda}A\Psi(x_1\partial_1\lambda + x_2\partial_2\lambda) - \int e^{\lambda}A\Psi,$$

$$\begin{aligned}&\int e^{\lambda}\left(x_1\frac{\chi'(r)}{r}b(\theta)\cos(\theta) + x_2\frac{\chi'(r)}{r}b(\theta)\sin(\theta)\right)\\ &= \int e^{\lambda}\chi'(r)b(\theta)(\cos^2(\theta)+\sin^2(\theta))\\ &= \left(\int \chi'(r)rdr\right)\left(\int b(\theta)d\theta\right) + \int (e^{\lambda}-1)\chi'(r)b(\theta)\end{aligned}$$

$$\begin{aligned}&\int e^{\lambda}\frac{\chi'(r)}{r^2}\frac{r_c}{\alpha}\left(x_1\left(b(\theta)\sin(\theta-\theta_c)\cos(\theta)\right)' + x_2\left(b(\theta)\sin(\theta-\theta_c)\sin(\theta)\right)'\right)\\ =& -\frac{r_c}{\alpha}\int e^{\lambda}\frac{\chi'(r)}{r}b(\theta)\sin(\theta-\theta_c)(-\cos(\theta)\sin(\theta)+\cos(\theta)\sin(\theta))\\ &- \frac{r_c}{\alpha}\int e^{\lambda}\partial_\theta\lambda\frac{\chi'(r)}{r}b(\theta)\sin(\theta-\theta_c)(\cos^2(\theta)+\sin^2(\theta))\\ =& -\frac{r_c}{\alpha}\int e^{\lambda}\partial_\theta\lambda\frac{\chi'(r)}{r}b(\theta)\sin(\theta-\theta_c).\end{aligned}$$

Therefore, in view of (2.4.17), (2.4.18) and (2.4.19), we have

$$\int f_1^{(1)} = \int f_2^{(1)} = \int x_1 f_1^{(1)} + x_2 f_2^{(2)} = 0$$

if and only if the quantities $\rho\cos(\eta), \rho\sin(\eta)$ and A are solutions of a linear system of the form

$$\begin{pmatrix} 1+O(\varepsilon) & O(\varepsilon) & O(\varepsilon)\\ O(\varepsilon) & 1+O(\varepsilon) & O(\varepsilon)\\ O(\varepsilon) & O(\varepsilon) & 1+O(\varepsilon)\end{pmatrix}\begin{pmatrix}\rho\cos(\eta)\\ \rho\sin(\eta)\\ A\end{pmatrix} = \begin{pmatrix}a_1\\ a_2\\ a_3\end{pmatrix},$$

where, since $\int \Psi = 2\pi$,

$$\begin{aligned}
a_1 &= \frac{1}{\pi}\int \dot{u}\partial_1 u + O(\varepsilon^2),\\
a_2 &= \frac{1}{\pi}\int \dot{u}\partial_2 u + O(\varepsilon^2),\\
a_3 &= -\frac{1}{2\pi}\int \dot{u}r\partial_r u + \frac{1}{2\pi}\left(\int \chi'(r)rdr\right)\int \widetilde{b}(\theta)d\theta + O(\varepsilon^2).
\end{aligned}$$

In the last equation we have used $\int b(\theta)d\theta = \int \widetilde{b}(\theta)d\theta$ to point out that this quantity does not depend on ρ, η. For $\varepsilon > 0$ small enough, this system is invertible, therefore we can find a unique triplet (ρ, η, A) in $\mathbb{R} \times \mathbb{S}^1 \times \mathbb{R}$ such that the three integrals are zero, and we have

$$\begin{aligned}
\rho\cos(\eta) &= \frac{1}{\pi}\int e^{\lambda}\dot{u}\partial_1 u + O(\varepsilon^2),\\
\rho\sin(\eta) &= \frac{1}{\pi}\int e^{\lambda}\dot{u}\partial_2 u + O(\varepsilon^2),\\
A &= -\frac{1}{2\pi}\int e^{\lambda}\dot{u}r\partial_r u + \frac{C(\chi)}{2\pi}\int \widetilde{b}(\theta)d\theta + O(\varepsilon^2).
\end{aligned}$$

This concludes the proof of Proposition (2.4.5)

2.5 The Lichnerowicz equation

Let H and τ be given by

$$\begin{aligned}
H &= e^{-\lambda}H^{(1)} + H^{(2)} + e^{-\lambda}H^{(3)},\\
\tau &= \tau^{(2)} + e^{-\lambda}\tau^{(3)} + A\Psi,
\end{aligned}$$

with ρ, η, A and $H^{(1)}$ given by Proposition (2.4.1). We recall $H^{(1)} = J\frac{\chi(r)}{r^2}N_\theta + \widetilde{H}^{(1)}$, and

$$|A| + |J| + |\rho| + \|e^{-\lambda}\widetilde{H}^{(1)}\|_{\mathcal{H}^1_{\delta+2}} \lesssim \varepsilon.$$

Proposition 2.5.1. *There exists a solution λ' of (2.3.2) which can be written uniquely under the form*

$$\lambda' = -\alpha'\chi(r)\ln(r) + r'_c\cos(\theta-\theta'_c)\frac{\chi(r)}{r} + \widetilde{\lambda}',$$

with $\widetilde{\lambda}' \in H^2_{\delta+1}$ and we have

$$\begin{aligned}
\alpha' &= \frac{1}{4\pi}\int \left(\dot{u}^2 + |\nabla u|^2\right) + O(\varepsilon^2),\\
r'_c\cos(\theta'_c) &= \frac{1}{4\pi}\int x_1\left(\dot{u}^2 + |\nabla u|^2\right) + O(\varepsilon^2),\\
r'_c\sin(\theta'_c) &= \frac{1}{4\pi}\int x_2\left(\dot{u}^2 + |\nabla u|^2\right) + O(\varepsilon^2),
\end{aligned}$$

and

$$\|\widetilde{\lambda}'\|_{H^2_{\delta+1}} \lesssim \left\|\dot{u}^2 + |\nabla u|^2\right\|_{H^0_{\delta+3}} + \varepsilon^2.$$

Proof. In order to apply Corollary (2.2.10) we have to check whether the right-hand side of (2.3.2) is in $H^0_{\delta+3}$. To estimate $|e^{-\lambda}\widetilde{H}^{(1)}|^2$, we use Proposition (2.2.4), which yields $|e^{-\lambda}\widetilde{H}^{(1)}|^2 \in H^0_{\delta+3}$ with

$$\left\| |e^{-\lambda}\widetilde{H}^{(1)}|^2 \right\|_{H^0_{\delta+3}} \lesssim \left\| e^{-\lambda}\widetilde{H}^{(1)} \right\|^2_{H^1_{\delta+2}} \lesssim \varepsilon^2. \tag{2.5.1}$$

To estimate terms of the form $\frac{|b|}{1+r}e^{-\lambda}|\widetilde{H}^{(1)}|$ we use Lemma (2.2.5). It yields

$$\left\| \frac{|b|}{1+r}e^{-2\lambda}|\widetilde{H}^{(1)}| \right\|_{H^0_{\delta+3}} \lesssim \|b\|_{W^{1,2}(\mathbb{S}^1)} \left\| e^{-\lambda}\widetilde{H}^{(1)} \right\|_{H^1_{\delta+2}} \lesssim \varepsilon^2. \tag{2.5.2}$$

To estimate terms of the form $\frac{|B'|}{1+r^2}e^{-\lambda}|\widetilde{H}^{(1)}|$ and $\frac{r_c}{\alpha}\frac{|b'|}{1+r^2}e^{-\lambda}|\widetilde{H}^{(1)}|$ we use Lemma (2.2.6), which yields

$$\begin{aligned} \left\| \frac{|B|+|B'|}{1+r^2}e^{-\lambda}|\widetilde{H}^{(1)}| \right\|_{H^0_{\delta+3}} &\lesssim \|B\|_{W^{1,2}(\mathbb{S}^1)} \left\| e^{-\lambda}\widetilde{H}^{(1)} \right\|_{H^1_{\delta+2}} \lesssim \varepsilon^2 \\ \left\| \frac{r_c}{\alpha}\frac{|b|+|b'|}{1+r^2}e^{-\lambda}|\widetilde{H}^{(1)}| \right\|_{H^0_{\delta+3}} &\lesssim \|b\|_{W^{1,2}(\mathbb{S}^1)} \left\| e^{-\lambda}\widetilde{H}^{(1)} \right\|_{H^1_{\delta+2}} \lesssim \varepsilon^2. \end{aligned} \tag{2.5.3}$$

In the same way, thanks to Proposition 2.2.4 and Lemma 2.2.6 we estimate

$$\|(A\Psi)^2\|_{H^0_{\delta+3}} \lesssim \varepsilon^2, \quad \left\| \frac{|b|}{1+r}A\Psi \right\|_{H^0_{\delta+3}} \lesssim \varepsilon^2, \quad ... \tag{2.5.4}$$

We can also estimate

$$\left\| \frac{r_c^2}{\alpha^2}\frac{(|b|+|b'|)^2}{1+r^4} \right\|_{H^0_{\delta+3}} \lesssim \varepsilon^2, \quad \left\| e^{-2\lambda}\frac{(|B|+|B'|)^2}{1+r^4} \right\|_{H^0_{\delta+3}} \lesssim \varepsilon^2, \quad ... \tag{2.5.5}$$

We now calculate

$$\begin{aligned} \frac{1}{2}|H|^2 - \frac{1}{4}\tau^2 = &\frac{1}{2}\left(-b(\theta)\frac{\chi(r)}{2r} - \left(\frac{r_c}{2\alpha}(b(\theta)\sin(\theta-\theta_c))' + e^{-\lambda}\frac{B'(\theta)}{2} \right)\frac{\chi(r)}{r^2} \right)^2 {M_\theta}^{ij} M_{\theta ij} \\ &- b(\theta)\frac{\chi(r)}{2r}\left(-\frac{r_c}{\alpha}b(\theta)\sin(\theta-\theta_c)\frac{\chi(r)}{r^2} + e^{-\lambda}(J-(1-\alpha)B(\theta))\frac{\chi(r)}{r^2} \right) {M_\theta}^{ij} N_{\theta ij} \\ &- \frac{1}{4}\left(b(\theta)\frac{\chi(r)}{r} + \left(\frac{r_c}{\alpha}(b(\theta)\sin(\theta-\theta_c))' + e^{-\lambda}B'(\theta) \right)\frac{\chi(r)}{r^2} \right)^2 + \widetilde{h}_1 \end{aligned}$$

where thanks to the estimates (2.5.1), (2.5.2), (2.5.3), (2.5.4), and (2.5.5), we have $\widetilde{h}_1 \in H^0_{\delta+3}$ with

$$\|\widetilde{h}_1\|_{H^0_{\delta+3}} \lesssim \varepsilon^2.$$

Since ${M_\theta}^{ij} M_{\theta ij} = 2$ and ${M_\theta}^{ij} N_{\theta ij} = 0$ we obtain

$$\frac{1}{2}|H|^2 - \frac{1}{4}\tau^2 = \widetilde{h}_1 \in H^0_{\delta+3}.$$

Consequently, we can solve (2.3.2) with Corollary (2.2.10), and the solution λ' can be written

$$\lambda' = -\alpha'\chi(r)\ln(r) + r_c\cos(\theta-\theta_c)\frac{\chi(r)}{r} + \widetilde{\lambda}',$$

with

$$\alpha' = \frac{1}{2\pi}\int\left(\frac{1}{2}\dot{u}^2+\frac{1}{2}|\nabla u|^2+\frac{1}{2}|H|^2-\frac{\tau^2}{4}\right) = \frac{1}{4\pi}\int\left(\dot{u}^2+|\nabla u|^2\right)+O(\varepsilon^2),$$
$$r_c\cos(\theta_c) = \frac{1}{2\pi}\int x_1\left(\frac{1}{2}\dot{u}^2+\frac{1}{2}|\nabla u|^2+\frac{1}{2}|H|^2-\frac{\tau^2}{4}\right) = \frac{1}{4\pi}\int x_1\left(\dot{u}^2+|\nabla u|^2\right)+O(\varepsilon^2),$$
$$r_c\sin(\theta_c) = \frac{1}{2\pi}\int x_2\left(\frac{1}{2}\dot{u}^2+\frac{1}{2}|\nabla u|^2+\frac{1}{2}|H|^2-\frac{\tau^2}{4}\right) = \frac{1}{4\pi}\int x_2\left(\dot{u}^2+|\nabla u|^2\right)+O(\varepsilon^2),$$

and $\widetilde{\lambda}' \in H^2_{\delta+1}$ such that

$$\|\widetilde{\lambda}'\|_{H^2_{\delta+1}} \lesssim \left\|\frac{1}{2}\dot{u}^2+\frac{1}{2}|\nabla u|^2+\frac{1}{2}|H|^2-\frac{\tau^2}{4}\right\|_{H^0_{\delta+3}} \lesssim \|\dot{u}^2+|\nabla u|^2\|_{H^0_{\delta+3}}+\varepsilon^2.$$

This concludes the proof of Proposition (2.5.1). □

2.6 Proof of Theorem (2.3.1)

We find it more convenient to perform the fixed point with the quantities (c_1, c_2) instead of r_c, θ_c. We recall the relation

$$(c_1, c_2) = r_c(\cos(\theta_c), \sin(\theta_c)).$$

We note X the Banach space

$$X = \mathbb{R}\times\mathbb{R}\times\mathbb{R}\times H^2_{\delta+1}$$

equipped with the norm

$$\|\lambda\|_X = \|(\alpha, c_1, c_2, \widetilde{\lambda})\|_X = |\alpha|+|c_1|+|c_2|+\|\widetilde{\lambda}\|_{H^2_{\delta+1}}.$$

We have constructed, for $\varepsilon > 0$ small enough, a map

$$F : X \to X$$

which maps $(\alpha, c_1, c_2, \widetilde{\lambda})$ satisfying

$$\|(\alpha, c_1, c_2, \widetilde{\lambda})\|_X = |\alpha|+|c_1|+|c_2|+\|\widetilde{\lambda}\|_{H^2_{\delta+1}} \lesssim \varepsilon,$$

and $\alpha \geq \frac{1}{2}\alpha_0$ where

$$\alpha_0 = \frac{1}{4\pi}\left(\int \dot{u}^2+|\nabla u|^2\right), \tag{2.6.1}$$

to $(\alpha', c_1', c_2', \widetilde{\lambda}')$ such that, for $\rho, \eta, A, H^{(1)}$ given by Proposition (2.4.1), if we note

$$\lambda = -\alpha\chi(r)\ln(r) + r_c\cos(\theta-\theta_c)\frac{\chi(r)}{r} + \widetilde{\lambda},$$
$$H = e^{-\lambda}H^{(1)} + H^{(2)} + e^{-\lambda}H^{(3)},$$
$$\tau = \tau^{(2)} + e^{-\lambda}\tau^{(3)} + A\Psi,$$

then H satisfies

$$\partial_i H_{ij} + H_{ij}\partial_i\lambda = -\dot{u}.\partial_j u + \frac{1}{2}\partial_j\tau - \frac{1}{2}\tau\partial_j\lambda,$$

and

$$\lambda' = -\alpha'\chi(r)\ln(r) + r'_c\cos(\theta-\theta'_c)\frac{\chi(r)}{r} + \widetilde{\lambda}'$$

is the solution of

$$\Delta\lambda' + \frac{1}{2}\dot{u}^2 + \frac{1}{2}|\nabla u|^2 + \frac{1}{2}|H|^2 - \frac{\tau^2}{4} = 0,$$

given by Proposition (2.5.1). Proposition (2.4.1) implies

$$|\rho| + |J| + |A| + \|\widetilde{H}^{(1)}\|_{\mathcal{H}^1_{\delta+2+\alpha}} \lesssim \varepsilon,$$

and Proposition (2.5.1) implies

$$|r'_c| + |\alpha'| + \|\widetilde{\lambda}'\|_{H^2_{\delta+1}} \lesssim \varepsilon.$$

In particular there exist C_0 such that

$$\|F(\alpha_0,0,0,0)\|_X = C_0\varepsilon.$$

Next we show that F is a contracting map in

$$B_X(0,2C_0\varepsilon) \cap \left\{\alpha \geq \frac{\alpha_0}{2}\right\}.$$

We consider, for $i=1,2$ $(\alpha_i,(c_1)_i,(c_2)_i,\widetilde{\lambda}_i)$ such that

$$\|(\alpha_i,(c_1)_i,(c_2)_i,\widetilde{\lambda}_i)\|_X \leq 2C_0\varepsilon, \quad \alpha_i \geq \frac{\alpha_0}{2}.$$

We note

$$(\alpha'_i,(c'_1)_i,(c'_2)_i,\widetilde{\lambda}'_i) = F(\alpha_i,(c_1)_i,(c_2)_i,\widetilde{\lambda}_i),$$

$$(r_c)_i(\cos(\theta_c)_i,\sin(\theta_c)_i) = ((c_1)_i,(c_2)_i), \quad (r'_c)_i(\cos(\theta'_c)_i,\sin(\theta'_c)_i) = ((c'_1)_i,(c'_2)_i).$$

Since $\alpha'_i = \alpha_0 + O(\varepsilon^2)$ we have for ε small enough

$$\alpha'_i \geq \frac{\alpha_0}{2}.$$

We note $\rho_i,\eta_i,A_i,J_i,\widetilde{H}^{(1)}_i$ the corresponding quantities given by Proposition (2.4.1). The proof of the following lemma is postponed to the end of this section.

Lemma 2.6.1. *We have the estimate*

$$|\alpha'_1-\alpha'_2| + |(c'_1)_1-(c'_1)_2| + |(c'_2)_1-(c'_2)_2| + \|\widetilde{\lambda}'_1-\widetilde{\lambda}'_2\|_{H^2_{\delta+1}} \lesssim \varepsilon\|\lambda_1-\lambda_2\|_X.$$

We are now in position to prove Theorem (2.3.1). Thanks to Lemma (2.6.1) there exists C such that

$$\|F(\lambda_1)-F(\lambda_2)\|_X \leq C\varepsilon\|\lambda_1-\lambda_2\|_X.$$

Consequently, by taking $\lambda_2 = (\alpha_0,0,0,0)$ we have

$$\forall\lambda \in B_X(0,2C_0\varepsilon) \cap \left\{\alpha \geq \frac{\alpha_0}{2}\right\}, \quad \|F(\lambda)-F(\alpha_0,0,0,0)\| \leq 2CC_0\varepsilon^2.$$

Therefore, if ε is small enough such that $C\varepsilon \leq 1$, the map F sends $B_X(0,2C_0\varepsilon)$ into itself. Moreover we already have noted that the condition $\alpha \geq \frac{\alpha_0}{2}$ is preserved by F for ε small enough. Finally, for $C\varepsilon < 1$ the map F is contracting, and the Picard fixed point Theorem yields the existence of a fixed point.

We now choose coordinates centered in the center of mass (c_1, c_2). For these coordinates, we have $r_c = 0$ and consequently

$$\begin{aligned}
\lambda &= -\alpha\chi(r)\ln(r) + \widetilde{\lambda}, \\
H &= -(\widetilde{b}(\theta) + \rho\cos(\theta-\eta))\frac{\chi(r)}{2r}M_\theta + e^{-\lambda}\frac{\chi(r)}{(r)^2}\left((J-(1-\alpha)B(\theta))N_\theta - \frac{B'(\theta)}{2}M_\theta\right) + \widetilde{H}, \\
\tau &= (\widetilde{b}(\theta) + \rho\cos(\theta-\eta))\frac{\chi(r)}{r} + e^{-\lambda}B'(\theta)\frac{\chi(r)}{r^2} + A\Psi,
\end{aligned}$$

The estimates of Propositions (2.4.1) and (2.5.1) complete the proof of Theorem (2.3.1).

To prove Lemma 2.6.1, we first prove the following two lemmas.

Lemma 2.6.2. *We have the estimate*

$$|\rho_1\cos(\eta_1) - \rho_2\cos(\eta_2)| + |\rho_1\sin(\eta_1) - \rho_2\sin(\eta_2)| + |A_1 - A_2| \lesssim \varepsilon\|\lambda_1 - \lambda_2\|_X.$$

Lemma 2.6.3. *We have the estimate*

$$\|e^{-\lambda_1}\widetilde{H}_1^{(1)} - e^{-\lambda_2}\widetilde{H}_1^{(1)}\|_{H^1_{\delta+2}} + |J_1 - J_2| \lesssim \varepsilon\|\lambda_1 - \lambda_2\|_X.$$

Proof of Lemma (2.6.2). The quantities $\rho_i\cos(\theta_i), \rho_i\sin(\theta_i), A_i$ are given by the expressions (2.4.17), (2.4.18), (2.4.19). Therefore we have

$$\begin{aligned}
&\pi(\rho_1\cos(\eta_1) - \rho_2\cos(\eta_2)) \\
&= \int_{\mathbb{R}^2} \left(e^{\lambda_1} - e^{\lambda_2}\right)\dot{u}\partial_1 u + e^{\lambda_1}A_1\Psi\partial_1\lambda_1 - e^{\lambda_2}A_2\Psi\partial_1\lambda_2 \\
&\quad - \int e^{\lambda_1}(h_1^{(2)})_1 - e^{\lambda_2}(h_1^{(2)})_2 + e^{\lambda_1}(h_1^{(3)})_1 - e^{\lambda_2}(h_1^{(3)})_2 \\
&\quad - \int (e^{\lambda_1} - e^{\lambda_2})\frac{\chi'(r)}{r}b_1(\theta)\cos(\theta)dx + (e^{\lambda_2} - 1)\frac{\chi'(r)}{r}(\rho_1\cos(\theta-\eta_1) - \rho_2\cos(\theta-\eta_2)) \\
&\quad + \int \frac{(r_c)_1}{\alpha_1}e^{\lambda_1}\frac{\chi'(r)}{r}b_1(\theta)\sin(\theta-(\theta_c)_1)\cos(\theta)\partial_\theta\lambda_1 - \frac{(r_c)_2}{\alpha_2}e^{\lambda_2}\frac{\chi'(r)}{r}b_2(\theta)\sin(\theta-(\theta_c)_2)\cos(\theta)\partial_\theta\lambda_2,
\end{aligned} \tag{2.6.2}$$

and a similar expression for $\rho_1\sin(\eta_1) - \rho_2\sin(\eta_2)$ and $A_1 - A_2$.

We estimate first $(h_j^{(2)})_1 - (h_j^{(2)})_2$, where the quantities $(h_j^{(2)})_i$ are defined by (2.4.3). We have

$$(h_j^{(2)})_1 - (h_j^{(2)})_2 = -\frac{1}{2}\tau_1^{(2)}\partial_j(\lambda_1-\lambda_2) + \frac{1}{2}(\tau_1^{(2)} - \tau_2^{(2)})\partial_j\lambda_2 - (H_{ij}^{(2)})_1\partial_i(\lambda_1-\lambda_2) + \left((H_{ij}^{(2)})_1 - (H_{ij}^{(2)})_2\right)\partial_i\lambda_2.$$

We calculate

$$\begin{aligned}
&\tau_1^{(2)} - \tau_2^{(2)} = (\rho_1\cos(\theta-\theta_1) - \rho_2\cos(\theta-\theta_2)) \\
&+ \left(\frac{(r_c)_1}{\alpha_1}((\rho_1\cos(\theta-\eta_1) + \widetilde{b}(\theta))\cos(\theta-(\theta_c)_1))' - \frac{(r_c)_2}{\alpha_2}((\rho_2\cos(\theta-\eta_2) + \widetilde{b}(\theta))\cos(\theta-(\theta_c)_2))'\right)\frac{\chi(r)}{r^2}.
\end{aligned}$$

We have a similar expression for $(H_{ij}^{(2)})_1 - (H_{ij}^{(2)})_2$. Therefore we have

$$\left\|(h_j^{(2)})_1 - (h_j^{(2)})_2\right\|_{H^0_{\delta+3}} \lesssim \varepsilon|\rho_1\cos(\eta_1) - \rho_2\cos(\eta_2)| + \varepsilon|\rho_1\sin(\eta_1) - \rho_2\sin(\eta_2)| + \varepsilon\|\lambda_1 - \lambda_2\|_X. \tag{2.6.3}$$

We now estimate $(h_j^{(3)})_1 - (h_j^{(3)})_2$, where the quantities $(h_j^{(3)})_i$ are defined by (2.4.4). The function $\tau^{(3)}$ does not depend on the index $i = 1, 2$. We calculate

$$H_1^{(3)} - H_2^{(3)} = \frac{\chi(r)}{r^2}(\alpha_1 - \alpha_2)B(\theta)N_\theta.$$

Therefore we obtain
$$\left\|(h_j^{(3)})_1 - (h_j^{(3)})_2\right\|_{H^0_{\delta+3}} \lesssim \varepsilon\|\lambda_1 - \lambda_2\|_X. \tag{2.6.4}$$

The estimates for the other terms of (2.6.2) are similar. Therefore (2.6.2), together with the estimates (2.6.3) and (2.6.4) yields

$$|\rho_1\cos(\eta_1)-\rho_2\cos(\eta_2)| \lesssim \varepsilon\left(|\rho_1\cos(\eta_1)-\rho_2\cos(\eta_2)| + |\rho_1\sin(\eta_1)-\rho_2\sin(\eta_2)| + |A_1-A_2|\right)+\varepsilon\|\lambda_1-\lambda_2\|_X.$$

Similarly we obtain

$$|\rho_1\sin(\eta_1)-\rho_2\sin(\eta_2)| \lesssim \varepsilon\left(|\rho_1\cos(\eta_1)-\rho_2\cos(\eta_2)| + |\rho_1\sin(\eta_1)-\rho_2\sin(\eta_2)| + |A_1-A_2|\right)+\varepsilon\|\lambda_1-\lambda_2\|_X$$

$$|A_1-A_2| \lesssim \varepsilon\left(|\rho_1\cos(\eta_1)-\rho_2\cos(\eta_2)| + |\rho_1\sin(\eta_1)-\rho_2\sin(\eta_2)| + |A_1-A_2|\right)+\varepsilon\|\lambda_1-\lambda_2\|_X$$

and consequently

$$|\rho_1\cos(\theta_1)-\rho_2\cos(\theta_2)| + |\rho_1\sin(\theta_1)-\rho_2\sin(\theta_2)| + |A_1-A_2| \lesssim \varepsilon\|\lambda_1-\lambda_2\|_X,$$

which concludes the proof of Lemma (2.6.2). □

Proof of Lemma (2.6.3). We compare first J_1 and J_2 thanks to the formula (2.4.8). We obtain

$$\begin{aligned}J_1 - J_2 = \frac{1}{2\pi}\int_{\mathbb{R}^2} &- \left(e^{\lambda_1}-e^{\lambda_2}\right)\dot{u}\partial_\theta u - (e^{\lambda_1}A_1 - e^{\lambda_2}A_2)\Psi\partial_\theta\lambda_1 - A_2 e^{\lambda_2}\Psi\partial_\theta(\lambda_1-\lambda_2)\\ &+ \frac{\rho_1(r_c)_1}{\alpha_1}\sin(\eta-(\theta_c)_1) - \frac{\rho_2(r_c)_2}{\alpha_2}\sin(\eta-(\theta_c)_2) + s.t.\end{aligned}$$

where the notation $s.t.$ stands for similar terms. Therefore, we obtain

$$|J_1-J_2| \lesssim \varepsilon\|\lambda_1-\lambda_2\|_X + |\rho_1\cos(\eta_1)-\rho_2\cos(\eta_2)| + |\rho_1\sin(\eta_1)-\rho_2\sin(\eta_2)| + |A_1-A_2|$$

and thanks to Lemma (2.6.2) we infer

$$|J_1-J_2| \lesssim \varepsilon\|\lambda_1-\lambda_2\|_X. \tag{2.6.5}$$

We now write the equation satisfied by $e^{-\lambda_1}\widetilde{H}_1^{(1)} - e^{-\lambda_2}\widetilde{H}_2^{(1)}$

$$\begin{aligned}&\partial_i\left(e^{-\lambda_1}\widetilde{H}_1^{(1)} - e^{-\lambda_2}\widetilde{H}_2^{(1)}\right)_{ij}\\ =&e^{-\lambda_1}(\widetilde{H}_1^{(1)})_{ij}\partial_j\lambda_1 - e^{-\lambda_2}(\widetilde{H}_2^{(1)})_{ij}\partial_j\lambda_2\\ &+ (e^{-\lambda_1}J_1 - e^{\lambda_2}J_2)\partial_i\left(\frac{\chi(r)}{r^2}N_\theta\right) + e^{\lambda_1}\partial_i(H_1^{(1)})_{ij} - e^{\lambda_2}\partial_i(H_2^{(1)})_{ij}\\ =&\left(e^{-\lambda_1}(\widetilde{H}_1^{(1)})_{ij} - e^{-\lambda_2}(\widetilde{H}_2^{(1)})_{ij}\right)\partial_j\lambda_1 + e^{-\lambda_2}(\widetilde{H}_2^{(1)})_{ij}\partial_j(\lambda_1-\lambda_2)\\ &+ (e^{-\lambda_1}J_1 - e^{\lambda_2}J_2)\partial_i\left(\frac{\chi(r)}{r^2}N_\theta\right) + (A_1-A_2)\partial_j\Psi + (h_j^{(2)})_1 - (h_j^{(2)})_2 + s.t.\end{aligned}$$

Consequently, Corollary (2.2.11) yields

$$\|e^{-\lambda_1}\widetilde{H}_1^{(1)} - e^{-\lambda_2}\widetilde{H}_2^{(1)}\|_{\mathcal{H}^1_{\delta+2}} \lesssim \varepsilon\|e^{-\lambda_1}\widetilde{H}_1^{(1)} - e^{-\lambda_2}\widetilde{H}_2^{(1)}\|_{\mathcal{H}^1_{\delta+2}} + |J_1-J_2| + \varepsilon\|\lambda_1-\lambda_2\|_X,$$

and thanks to (2.6.5)
$$\|e^{-\lambda_1}\widetilde{H}_1^{(1)} - e^{-\lambda_2}\widetilde{H}_2^{(1)}\|_{\mathcal{H}^1_{\delta+2}} \lesssim \varepsilon\|\lambda_1-\lambda_2\|_X,$$

which concludes the proof of Lemma (2.6.3). □

Proof of Lemma (2.6.1). In view of (2.3.2) we have

$$\Delta(\lambda_1' - \lambda_2') = -\frac{1}{2}|H_1|^2 + \frac{1}{4}\tau_1^2 + \frac{1}{2}|H_2|^2 - \frac{1}{4}\tau_2^2.$$

The right-hand side is in $H^0_{\delta+3}$ and satisfies

$$\begin{aligned}
&\left\| \frac{1}{2}|H_1|^2 - \frac{1}{4}\tau_1^2 - \frac{1}{2}|H_2|^2 - \frac{1}{4}\tau_2^2 \right\|_{H^0_{\delta+3}} \\
&\lesssim \varepsilon \left(\|e^{-\lambda_1}\widetilde{H}_1^{(1)} - e^{-\lambda_2}\widetilde{H}_2^{(1)}\|_{H^1_{\delta+2}} + |J_1 - J_2| \right) + \varepsilon\|\lambda_1 - \lambda_2\|_X \\
&\lesssim \varepsilon\|\lambda_1 - \lambda_2\|_X,
\end{aligned}$$

where we have used Lemma 2.6.3 in the last inequality. Therefore Corollary (2.2.10) allows us to write

$$\lambda_1' - \lambda_2' = -(\alpha_1' - \alpha_2')\chi(r)\ln(r) + ((r_c')_1\cos(\theta - (\theta_c')_1) - (r_c')_2\cos(\theta - (\theta_c')_2))\frac{\chi(r)}{r} + \widetilde{\lambda}_1' - \widetilde{\lambda}_2',$$

with

$$|\alpha_1' - \alpha_2'| + |(c_1')_1 - (c_1')_2| + |(c_2')_1 - (c_2')_2| + \|\widetilde{\lambda}_1' - \widetilde{\lambda}_2'\|_{H^2_{\delta+1}} \lesssim \varepsilon\|\lambda_1 - \lambda_2\|_X.$$

This concludes the proof of Lemma (2.6.1). □

Chapitre 3

Stability in exponential time in the asymptotically flat case

3.1 Introduction

In this chapter, we address the quasi stability of the Minkowski solution to the Einstein vacuum equations with a translation space-like Killing field. In the presence of a translation space-like Killing field, the $3+1$ Einstein vacuum equations reduces to the following system

$$\left\{\begin{array}{l}\Box_g \varphi = 0,\\ R_{\mu\nu} = \partial_\mu \varphi \partial_\nu \varphi.\end{array}\right. \tag{3.1.1}$$

This system has been studied by Choquet-Bruhat and Moncrief in [11] (see also [8]) in the case of a space-time of the form $\Sigma \times \mathbb{S}^1 \times \mathbb{R}$, where Σ is a compact two dimensional manifold of genus $G \geq 2$, and $\mathbb{R}$ is the time axis, with a space-time metric independent of the coordinate on $\mathbb{S}^1$. They prove the existence of global solutions corresponding to the perturbation of a particular expanding universe. This symmetry has also been studied in [3], with an additional rotation symmetry.

In this paper, we consider a space-time of the form $\mathbb{R}^2 \times \mathbb{R}_{x^3} \times \mathbb{R}_t$, for which ∂_3 is a Killing vector field. Minkowski space-times can be seen as a trivial solution of Einstein vacuum equations with this symmetry. The question we address in this paper is the stability of the Minkowski solution in this framework.

In the $3+1$ vacuum case, the stability of Minkowski space-time has been proven in the celebrated work of Christodoulou and Klainerman in [13] in the maximal foliation. It has then been proven by Lindblad and Rodnianski using harmonic gauge in [40]. Their proof extends also to Einstein equations coupled to a scalar field. In this work we will use wave coordinates.

3.1.1 Einstein equations in wave coordinates

Wave coordinates (x^α) are required to satisfy $\Box_g x^\alpha = 0$. In these coordinates (3.1.1) reduces to the following system of quasilinear wave equations

$$\left\{\begin{array}{l}\Box_g \varphi = 0,\\ \Box_g g_{\mu\nu} = -\partial_\mu \varphi \partial_\nu \varphi + P_{\mu\nu}(\partial g, \partial g),\end{array}\right. \tag{3.1.2}$$

where $P_{\mu\nu}$ is a quadratic form. To understand the difficulty, let us first recall known results in $3+1$ dimensions. In $3+1$ dimensions, a semi linear system of wave equations of the form

$$\Box u^i = P^i(\partial u^j, \partial u^k)$$

is critical in the sense that if there isn't enough structure, the solutions might blow up in finite time (see the counter examples by John [31]). However, if the right-hand side satisfies the null condition, introduced by Klainerman in [32], the system admits global solutions. This condition requires that P^i be linear combinations of the following forms

$$Q_0(u,v) = \partial_t u \partial_t v - \nabla u.\nabla v, \quad Q_{\alpha\beta}(u,v) = \partial_\alpha u \partial_\beta v - \partial_\alpha v \partial_\beta u.$$

In three dimensions, Einstein equations written in wave coordinates do not satisfy the null condition. However, this is not a necessary condition to obtain global existence. An example is provided by the system

$$\begin{cases} \Box \varphi_1 = 0, \\ \Box \varphi_2 = (\partial_t \varphi_1)^2. \end{cases} \tag{3.1.3}$$

The non-linearity does not have the null structure, but thanks to the decoupling there is nevertheless global existence. In [39], Lindblad and Rodnianski showed that the non linear terms in Einstein equations in wave coordinates consists of a linear combination of null forms with an underlying structure of the form (3.1.3). They used the wave condition to obtain better decay for some coefficients of the metric. However the decay is slower than for the solution of the wave equation. An example of a quasilinear scalar wave equation admitting global existence without the null condition, but with a slower decay is also studied by Alinhac in [2] and Lindblad in [38]. In [39], Lindblad and Rodnianski introduced the notion of weak-null structure, which gather all these examples.

In $2+1$ dimensions, to show global existence, one has to be careful with both quadratic and cubic terms. Quasilinear scalar wave equations in $3+1$ dimensions have been studied by Alinhac in [1]. He shows global existence for a quasilinear equation of the form

$$\Box u = g^{\alpha\beta}(\partial u)\partial_\alpha \partial_\beta u,$$

if the quadratic and cubic terms in the right-hand side satisfy the null condition. Global existence for a semi-linear wave equation with the quadratic and cubic terms satisfying the null condition has been shown by Godin in [26] using an algebraic trick to remove the quadratic terms, which does however not extend to systems. The global existence in the case of systems of semi-linear wave equations with the null structure has been shown by Hoshiga in [28]. It requires the use of $L^\infty - L^\infty$ estimates for the inhomogeneous wave equations, introduced in [35].

To show the quasi global existence for our system in wave coordinates, it will therefore be necessary to exhibit structure in quadratic and cubic terms. However, as for the vacuum Einstein equations in $3+1$ dimension in wave coordinates, our system does not satisfy the null structure. It will in particular be important to understand what happens for a system of the form (3.1.3) in $2+1$ dimensions. For such a system, standard estimates only give an L^∞ bound for φ_2, without decay. Moreover, the growth of the energy of φ_2 is like $\sqrt{t}$.

One can easily imagine that with more intricate a coupling than for (3.1.3), it will be very difficult to prove stability without decay for φ_2. To obtain a more useful estimate, the idea will be to exploit more precisely the fact that φ_1 also satisfies a wave equation. To understand how this might help, we will look at special solutions of vacuum Einstein equations with a translation space-like Killing field : Einstein-Rosen waves. These solutions have been discovered by Beck (see [5], and also [3] and [6] for a mathematical description).

3.1.2 Einstein-Rosen waves

Einstein-Rosen waves are solutions of vacuum Einstein equations with two space-like orthogonal Killing fields : ∂_3 and ∂_θ. The $3+1$ metric can be written

$$\mathbf{g} = e^{2\varphi}(dx^3)^2 + e^{2(a-\varphi)}(-dt^2 + dr^2) + r^2 e^{-2\varphi} r^2 d\theta^2.$$

The reduced equations

$$\begin{cases} R_{\mu\nu} = \partial_\mu \varphi \partial_\nu \varphi, \\ \Box_g \varphi = 0, \end{cases}$$

can be written in this setting

$$R_{tt} = \partial_r^2 a - \partial_t^2 a + \frac{1}{r}\partial_r a = 2(\partial_t\varphi)^2, \tag{3.1.4}$$
$$R_{rr} = -\partial_r^2 a + \partial_t^2 a + \frac{1}{r}\partial_r a = 2(\partial_r\varphi)^2,$$
$$R_{tr} = \frac{1}{r}\partial_t a = 2\partial_t\varphi\partial_r\varphi.$$

The equation for φ can be written, since φ is radial

$$e^{2a}\Box_g\varphi = -\partial_t^2\varphi + \partial_r^2\varphi + \frac{1}{r}\partial_r\varphi = 0,$$

where g is the metric

$$g = e^{2a}(-dt^2 + dr^2) + r^2 d\theta^2.$$

The equation for φ decouples from the equations for the metric. Therefore we can solve the flat wave equation $\Box\varphi = 0$, with initial data $(\varphi, \partial_t\varphi)|_{t=0} = (\varphi_0, \varphi_1)$ and then solve the Einstein equations, which reduces to

$$\partial_r a = r\left((\partial_r\varphi)^2 + (\partial_t\varphi)^2\right), \tag{3.1.5}$$

with the boundary condition $\varphi|_{r=0} = 0$ in order to have a smooth solution. Since $\Box\varphi = 0$, if (φ_0, φ_1) have enough decay, we have the following decay estimate for φ

$$|\partial\varphi(r,t)| \lesssim \frac{1}{\sqrt{1+t+r}(1+|t-r|)^{\frac{3}{2}}}.$$

Therefore since

$$a = \int_0^R r\left((\partial_r\varphi)^2 + (\partial_t\varphi)^2\right) dr$$

we have

$$|a| \lesssim \frac{1}{(1+|r-t|)^2}, \; for\; r < t,$$
$$|a - E(\varphi)| \lesssim \frac{1}{(1+|r-t|)^2}, \; for\; r > t,$$

where the energy

$$E(\varphi) = \int_0^\infty r\left((\partial_r\varphi)^2 + (\partial_t\varphi)^2\right) dr$$

does not depend on t. For $r > t$, we have $a \sim E(\gamma)$ and hence is only bounded. In particular, the metric

$$e^{2a}dr^2 + r^2 d\theta^2$$

exhibits an angle at space-like infinity, that is to say the circles of radius r have a perimeter growth of $e^{-E(\varphi)}2\pi r$ instead of $2\pi r$. However, in the interior, the decay we get is far better than the one we could have found with standard estimates, if we had used (3.1.4) instead of (3.1.5).

3.1.3 The background metric

We would like to adapt the analysis of Section 3.1.2 in the case where we only assume one Killing field (i.e. in the case where ∂_3 is Killing but not ∂_θ). Assume that

$$a = \int_0^R r\left((\partial_r\varphi)^2 + (\partial_t\varphi)^2\right) dr$$

is still an approximate solution of (3.1.3), which will appear to be true in Section 3.7. As in this case φ also depends on θ, we will have

$$\lim_{R\to\infty} a(t,R,\theta) = \int_0^\infty r\left((\partial_r\varphi)^2 + (\partial_t\varphi)^2\right)dr = b(t,\theta).$$

Note that we have to be careful with the dependence on θ. The metric

$$e^{2b(\theta)}(-dt^2+dr^2)+r^2d\theta^2$$

is no longer a Ricci flat metric when b depends on θ. Consequently it is not a good guess for the behavior at infinity of our metric solution g. A good candidate should be Ricci flat in the region $r>t$. Indeed if we considered compactly supported initial data for φ, by finite speed propagation, φ should intuitively be supported in the region $r<t$. Consequently, the equation

$$R_{\mu\nu} = \partial_\mu\varphi\partial_\nu\varphi$$

implies that g should be Ricci flat for $r>t$. Consequently, we are yield to consider the following family of space-time metrics

$$g_b = -dt^2 + dr^2 + (r+\chi(q)b(\theta)q)^2d\theta^2 + J(\theta)\chi(q)dqd\theta, \tag{3.1.6}$$

where (r,θ) are polar coordinates, $q=r-t$ and χ is a cut-off function such that $\chi(q)=0$ for $q<1$ and $\chi(q)=1$ for $q>2$. In the coordinates $s=r+t,q,\theta$, a tedious calculation yield that all the Ricci coefficients are zero except

$$\begin{aligned}(R_b)_{qq} &= -\frac{b(\theta)\partial_q^2(q\chi(q))}{r+b(\theta)q\chi(q)} + \frac{q\chi(q)\chi'(q)J(\theta)\partial_\theta b}{(r+b(\theta)q\chi(q))^3} + \frac{J(\theta)^2\chi(q)\chi'(q)}{4(r+b(\theta)q\chi(q))^3} - \frac{\chi'(q)\partial_\theta J(\theta)}{(r+b(\theta)q\chi(q))^2},\\ &= -\frac{b(\theta)\partial_q^2(q\chi(q))}{r} + O\left(\frac{C(b,b',J,J')\mathbb{1}_{1<q<2}}{r^2}\right),\end{aligned} \tag{3.1.7}$$

$$(R_b)_{q\theta} = -\frac{J(\theta)\chi'(q)}{2(r+b(\theta)q\chi(q))} = O\left(\frac{C(b,J)\mathbb{1}_{1<q<2}}{r}\right). \tag{3.1.8}$$

Therefore, the metrics g_b are Ricci flat in the region $r>t+2$. We will see in the next section that they are compatible with the initial data for g given by the constraint equations.

This choice of background metric will force us to work in generalized wave coordinates, instead of usual wave coordinates. Indeed, for the metric g_b defined by (3.1.6), the coordinates (t,x_1,x_2) are not wave coordinates, not even asymptotically. The generalized wave coordinate condition reads, for g of the form $g=g_b+\widetilde{g}$

$$g^{\lambda\beta}\Gamma^\alpha_{\lambda\beta} = H_b^\alpha$$

where H_b^α is defined by

$$H_b^\alpha = \bar{H}_b^\alpha + F^\alpha, \tag{3.1.9}$$

where $\bar{H}_b^\alpha$ is defined by

$$\bar{H}_b^\alpha = g_b^{\lambda\beta}(\Gamma_b)^\alpha_{\lambda\beta} \tag{3.1.10}$$

and F^α is defined by the sum of the crossed terms of the form $\widetilde{g}\frac{\partial_\theta}{r}g_b$ in $g^{\lambda\beta}\Gamma^\alpha_{\lambda\beta} - \bar{H}^\alpha_b$. The reason of this choice for F^α will be explained in next section, in the proof of Theorem 3.1.3.

The form of (3.1.1) in generalized wave coordinates is given by (3.2.1) .

3.1.4 The initial data

In this section, we will explain how to choose the initial data for φ and g. We will note i, j the space-like indices and α, β the space-time indices.

We will work in weighted Sobolev spaces.

Definition 3.1.1. *Let $m \in \mathbb{N}$ and $\delta \in \mathbb{R}$. The weighted Sobolev space $H^m_\delta(\mathbb{R}^n)$ is the completion of C^∞_0 for the norm*

$$\|u\|_{H^m_\delta} = \sum_{|\beta|\leq m} \|(1+|x|^2)^{\frac{\delta+|\beta|}{2}} D^\beta u\|_{L^2}.$$

The weighted Hölder space C^m_δ is the complete space of m-times continuously differentiable functions with norm

$$\|u\|_{C^m_\delta} = \sum_{|\beta|\leq m} \|(1+|x|^2)^{\frac{\delta+|\beta|}{2}} D^\beta u\|_{L^\infty}.$$

Let $0<\alpha<1$. The Hölder space $C^{m+\alpha}_\delta$ is the the complete space of m-times continuously differentiable functions with norm

$$\|u\|_{C^{m+\alpha}_\delta} = \|u\|_{C^m_\delta} + \sup_{x\neq y,\ |x-y|\leq 1} \frac{|\partial^m u(x) - \partial^m u(y)|(1+|x|^2)^{\frac{\delta}{2}}}{|x-y|^\alpha}.$$

We recall the Sobolev embedding with weights (see for example [8], Appendix I).

Proposition 3.1.2. *Let $s, m \in \mathbb{N}$. We assume $s>1$. Let $\beta \leq \delta+1$ and $0<\alpha<min(1, s-1)$. Then, we have the continuous embedding*

$$H^{s+m}_\delta(\mathbb{R}^2) \subset C^{m+\alpha}_\beta(\mathbb{R}^2).$$

Let $0<\delta<1$. The initial data (φ_0, φ_1) for $(\varphi, \partial_t\varphi)|_{t=0}$ are freely given in $H^{N+1}_\delta \times H^N_{\delta+1}$ with $0<\delta<1$. However the initial data for $(g_{\mu\nu}, \partial_t g_{\mu\nu})$ cannot be chosen arbitrarily.

- The induced metric and second fundamental form $(\bar{g}, K)$ must satisfy the constraint equations.
- The generalized wave coordinates condition must be satisfied at $t=0$.

Moreover, we want to prescribe the asymptotic behaviour for g : we want it to be asymptotic to g_b, where $b(\theta)$ is arbitrarily prescribed, except for its components in 1, $\cos(\theta)$ and $sin(\theta)$.

We recall the constraint equations. First we write the metric g in the form

$$g = -N^2(dt)^2 + \bar{g}_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt),$$

where the scalar function N is called the lapse, the vector field β is called the shift and $\bar{g}$ is a Riemannian metric on $\mathbb{R}^2$.

We consider the initial space-like surface $\mathbb{R}^2 = \{t=0\}$. We will use the notation

$$\partial_0 = \partial_t - \mathcal{L}_\beta,$$

where $\mathcal{L}_\beta$ is the Lie derivative associated to the vector field β. With this notation, we have the following expression for the second fundamental form of $\mathbb{R}^2$

$$K_{ij} = -\frac{1}{2N}\partial_0 g_{ij}.$$

We will use the notation

$$\tau = g^{ij} K_{ij}$$

for the mean curvature. We also introduce the Einstein tensor

$$G_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2}Rg_{\alpha\beta},$$

where R is the scalar curvature $R = g^{\alpha\beta}R_{\alpha\beta}$. The constraint equations are given by

$$G_{0j} \equiv N(\partial_j\tau - D^iK_{ij}) = \partial_0\varphi\partial_j\varphi,\ j = 1, 2, \tag{3.1.11}$$

$$G_{00} \equiv \frac{N^2}{2}(\bar{R} - |K|^2 + \tau^2) = (\partial_0\varphi)^2 - \frac{1}{2}g_{00}g^{\alpha\beta}\partial_\alpha\varphi\partial_\beta\varphi, \tag{3.1.12}$$

where D and $\bar{R}$ are respectively the covariant derivative and the scalar curvature associated to $\bar{g}$. The following result, proven in Appendix 3.12.1, gives us the initial data we need.

Theorem 3.1.3. *Let $0 < \delta < 1$. Let $(\varphi_0, \varphi_1) \in H^{N+1}_\delta(\mathbb{R}^2) \times H^N_{\delta+1}(\mathbb{R}^2)$ and $\widetilde{b}(\theta) \in W^{N,2}(\mathbb{S}^1)$ such that*

$$\int \widetilde{b}d\theta = \int \widetilde{b}\cos(\theta)d\theta = \int \widetilde{b}\sin(\theta)d\theta = 0.$$

We assume

$$\|\varphi_0\|_{H^{N+1}_\delta} + \|\varphi_1\|_{H^N_{\delta+1}} \lesssim \varepsilon, \quad \|\widetilde{b}\|_{W^{N,2}} \lesssim \varepsilon^2.$$

If $\varepsilon > 0$ is small enough, there exists $b_0, b_1, b_2 \in \mathbb{R}\times\mathbb{R}\times\mathbb{S}^1$, $J \in W^{N,2}(\mathbb{S}^1)$ and

$$(g_{\alpha\beta})_0, (g_{\alpha\beta})_1 \in H^{N+1}_\delta \times H^N_{\delta+1}$$

such that the initial data for g given by

$$g = g_b + g_0,\ \partial_t g = \partial_t g_b + g_1,$$

where g_b is defined by (3.1.6) with

$$b(\theta) = b_0 + b_1\cos(\theta) + b_2\sin(\theta) + \widetilde{b}(\theta),$$

are such that

- *$g_{ij}, K_{ij} = \mathcal{L}_\beta g_{ij}$ satisfy the constraint equations (3.1.11) and (3.1.12).*
- *the following generalized wave coordinates condition is satisfied at $t = 0$*

$$g^{\lambda\beta}\Gamma^\alpha_{\lambda\beta} = g_b^{\lambda\beta}(\Gamma_b)^\alpha_{\lambda\beta} + F^\alpha,$$

where F^α is the sum of all the crossed term of the form $g_0\frac{\partial_\theta}{r}g_b$ in $g^{\lambda\beta}\Gamma^\alpha_{\lambda\beta} - g_b^{\lambda\beta}(\Gamma_b)^\alpha_{\lambda\beta}$.

Moreover, we have the estimates

$$\|J\|_{W^{N,2}(\mathbb{S}^1)} + \|g_0\|_{H^{N+1}_\delta} + \|g_1\|_{H^N_{\delta+1}} \lesssim \varepsilon^2,$$

$$b_0 = \frac{1}{4\pi}\int\left(\dot{\varphi}^2 + |\nabla\varphi|^2\right) + O(\varepsilon^4),$$

$$b_1 = \frac{1}{\pi}\int\dot{\varphi}\partial_1\varphi + O(\varepsilon^4),$$

$$b_2 = \frac{1}{\pi}\int\dot{\varphi}\partial_2\varphi + O(\varepsilon^4),$$

Let us make a remark on the choice of F

Remark 3.1.4. *The initial data $\partial_t\widetilde{g}_{00}$ and $\partial_t\widetilde{g}_{0i}$ are constructed so the generalized wave coordinate condition is satisfied at $t = 0$. The choice of F is here to prevent terms of the form $\widetilde{g}\partial_U(g_b)$ in this condition, and therefore allow us to have*

$$\partial_t\widetilde{g}_{00}, \partial_t\widetilde{g}_{0i} \in H^N_{\delta+1}.$$

Before stating our main result, we will recall some notations and basic tools in the study of wave equations.

3.1.5 Some basic tools

Coordinates and frames

- We note x^α the standard space-time coordinates, with $t = x^0$. We note (r, θ) the polar space-like coordinates, and $s = t + r$, $q = r - t$ the null coordinates. The associated one-forms are

$$ds = dt + dr, \quad dq = dr - dt,$$

and the associated vector fields are

$$\partial_s = \frac{1}{2}(\partial_t + \partial_r), \quad \partial_q = \frac{1}{2}(\partial_r - \partial_t).$$

- We note ∂ the space-time derivatives, ∇ the space-like derivatives, and by $\bar{\partial}$ the derivatives tangent to the future directed light-cone in Minkowski, that is to say $\partial_t + \partial_r$ and $\frac{\partial_\theta}{r}$.

- We introduce the null frame $L = \partial_t + \partial_r$, $\underline{L} = \partial_t - \partial_r$, $U = \frac{\partial_\theta}{r}$. In this frame, the Minkowski metric takes the form

$$m_{L\underline{L}} = -2,\ m_{UU} = 1,\ m_{LL} = m_{\underline{L}\underline{L}} = m_{LU} = m_{\underline{L}U} = 0.$$

The collection $\mathcal{T} = \{U, L\}$ denotes the vector fields of the frame tangent to the light-cone, and the collection $\mathcal{V} = \{U, L, \underline{L}\}$ denotes the full null frame.

The flat wave equation Let φ be a solution of

$$\begin{cases} \Box\varphi = 0, \\ (\varphi, \partial_t\varphi)|_{t=0} = (\varphi_0, \varphi_1). \end{cases} \tag{3.1.13}$$

The following proposition establishes decay for the solutions of $2+1$ dimensional flat wave equation.

Proposition 3.1.5 (Proposition 2.1 in [36]). *Let $\mu > \frac{1}{2}$. We have the estimate*

$$|\varphi(x,t)| \lesssim M_\mu(\varphi_0, \varphi_1) \frac{(1 + |t-r|)^{[1-\mu]_+}}{\sqrt{1+t+r}\sqrt{1+|t-r|}}$$

where

$$M_\mu(\varphi_0, \varphi_1) = \sup_{y \in \mathbb{R}^2} (1+|y|)^\mu |\varphi_0(y)| + (1+|y|)^{\mu+1}(|\varphi_1(y)| + |\nabla\varphi_0(y)|)$$

and where we used the notation $A^{[\alpha]_+} = A^{\max(\alpha,0)}$ if $\alpha \neq 0$ and $A^{[0]_+} = \ln(A)$.

Minkowski vector fields We will rely in a crucial way on the Klainerman vector field method. We introduce the following family of vector fields

$$\mathcal{Z} = \{\partial_\alpha, \Omega_{\alpha\beta} = -x_\alpha\partial_\beta + x_\beta\partial_\alpha, S = t\partial_t + r\partial_r\},$$

where $x_\alpha = m_{\alpha\beta}x^\beta$. These vector field satisfy the commutation property

$$[\Box, Z] = C(Z)\Box,$$

where

$$C(Z) = 0,\ Z \neq S, \quad C(S) = 2.$$

Moreover some easy calculations give

$$\begin{aligned}
\partial_t + \partial_r &= \frac{S + \cos(\theta)\Omega_{0,1} + \sin(\theta)\Omega_{0,2}}{t+r}, \\
\frac{1}{r}\partial_\theta &= \frac{\Omega_{1,2}}{r} = \frac{\cos(\theta)\Omega_{0,2} - \sin(\theta)\Omega_{0,1}}{t}, \\
\partial_t - \partial_r &= \frac{S - \cos(\theta)\Omega_{0,1} - \sin(\theta)\Omega_{0,2}}{t-r}.
\end{aligned}$$

With this calculations, and the commutations properties in the region $-\frac{t}{2} \leq r \leq 2t$

$$[Z, \partial] \sim \partial, \; [Z, \bar{\partial}] \sim \bar{\partial},$$

we obtain

$$|\partial^k \bar{\partial}^l u| \leq \frac{1}{(1+|q|)^k (1+s)^l} |Z^{k+l} u|, \tag{3.1.14}$$

where here and in the rest of the paper, $Z^I u$ denotes any product of I of the vector fields of $\mathcal{Z}$. Estimates (3.1.14) and Proposition 3.1.5 yield

Corollary 3.1.6. *Let φ be a solution of* (3.1.13). *We have the estimate*

$$|\partial^k \bar{\partial}^l \varphi(x,t)| \lesssim M_\mu^{k+l}(\varphi_0, \varphi_1) \frac{(1+|t-r|)^{[1-\mu]_+}}{(1+t+r)^{l+\frac{1}{2}} (1+|t-r|)^{k+\frac{1}{2}}}$$

where

$$M_\mu^j(\varphi_0, \varphi_1) = \sup_{y \in \mathbb{R}^2} (1+|y|)^{\mu+j} |\nabla^s \varphi_0(y)| + (1+|y|)^{\mu+1+j} (|\nabla^s \varphi_1(y)| + |\nabla^{1+j} \varphi_0(y)|).$$

Weighted energy estimate We consider a weight function $w(q)$, where $q = r - t$, such that $w'(q) > 0$ and

$$\frac{w(q)}{(1+|q|)^{1+\mu}} \lesssim w'(q) \lesssim \frac{w(q)}{1+|q|},$$

for some $0 < \mu < \frac{1}{2}$.

Proposition 3.1.7. *We assume that $\Box \varphi = f$. Then we have*

$$\begin{aligned}
&\frac{1}{2}\partial_t \int w(q) \left((\partial_t \varphi)^2 + |\nabla \varphi|^2 \right) + \frac{1}{2} \int w'(q) \left((\partial_s \varphi)^2 + \left(\frac{\partial_\theta u}{r} \right)^2 \right) \\
&\lesssim \int w(q) |f \partial_t \varphi|.
\end{aligned}$$

For the proof of Proposition 3.1.7, we refer to the proof of Proposition 3.9.1 which is the quasilinear equivalent of Proposition 3.1.7.

Weighted Klainerman-Sobolev inequality The following proposition allows us to obtain L^∞ estimates from the energy estimates. It is proved in Appendix 3.12.5. The proof is inspired from the corresponding $3+1$ dimensional proposition (Proposition 14.1 in [40]).

Proposition 3.1.8. *We denote by v any of our weight functions. We have the inequality*

$$|f(t,x) v^{\frac{1}{2}}(|x|-t)| \lesssim \frac{1}{\sqrt{1+t+|x|}\sqrt{1+||x|-t|}} \sum_{|I| \leq 2} \|v^{\frac{1}{2}}(.-t) Z^I f\|_{L^2}.$$

Weighted Hardy inequality If u is solution of $\Box u = f$, the energy estimate allows us to estimate the L^2 norm of ∂u. To estimate the L^2 norm of u, we will use a weighted Hardy inequality.

Proposition 3.1.9. *Let $\alpha < 1$ and $\beta > 1$. We have, with $q = r - t$*

$$\left\| \frac{v(q)^{\frac{1}{2}}}{(1+|q|)} f \right\|_{L^2} \lesssim \|v(q)^{\frac{1}{2}} \partial_r f\|_{L^2},$$

where

$$v(q) = (1+|q|)^{\alpha}, \; for \; q < 0,$$
$$v(q) = (1+|q|)^{\beta}, \; for \; q > 0.$$

This is proven in Appendix 3.12.4. The proof is inspired from the $3+1$ dimensional analogue (Lemma 13.1 in [40]).

$L^\infty - L^\infty$ estimate With the condition $w'(q) > 0$ for the energy inequality, we are not allowed to take weights of the form $(1+|q|)^{\alpha}$, with $\alpha > 0$ in the region $q < 0$. Therefore, Klainerman-Sobolev inequality cannot give us more than the estimate

$$|\partial u| \lesssim \frac{1}{\sqrt{1+|q|}\sqrt{1+s}},$$

in the region $q < 0$, for a solution of $\Box u = f$. However, we know that for suitable initial data, the solution of the wave equation $\Box u = 0$ satisfies

$$|u| \lesssim \frac{1}{\sqrt{1+|q|}\sqrt{1+s}}, \; |\partial u| \lesssim \frac{1}{(1+|q|)^{\frac{3}{2}}\sqrt{1+s}}.$$

To recover some of this decay we will use the following proposition

Proposition 3.1.10. *Let u be a solution of*

$$\begin{cases} \Box u = F, \\ (u, \partial_t u)|_{t=0} = (0,0). \end{cases}$$

For $\mu > \frac{3}{2}, \nu > 1$ we have the following $L^\infty - L^\infty$ estimate

$$|u(t,x)|(1+t+|x|)^{\frac{1}{2}} \leq C(\mu,\nu) M_{\mu,\nu}(F)(1+|t-|x||)^{-\frac{1}{2}+[2-\mu]_+},$$

where

$$M_{\mu,\nu}(F) = \sup (1+|y|+s)^{\mu}(1+|s-|y||)^{\nu} F(y,s),$$

and where we used the convention $A^{[\alpha]_+} = A^{\max(\alpha,0)}$ if $\alpha \neq 0$ and $A^{[0]_+} = \ln(A)$.

This is proven in Appendix 3.12.3. This inequality has been introduced by Kubo and Kubota in [35].

An integration lemma The following lemma will be used many times in the proof of Theorem 3.1.12, to obtain estimates for u when we only have estimates for ∂u.

Lemma 3.1.11. *Let $\alpha, \beta, \gamma \in \mathbb{R}$ with $\beta < -1$. We assume that the function $u : \mathbb{R}^{2+1} \to \mathbb{R}$ satisfies*

$$|\partial u| \lesssim (1+s)^\gamma (1+|q|)^\alpha, \ for\ q<0, \quad |\partial u| \lesssim (1+s)^\gamma (1+|q|)^\beta \ for\ q>0,$$

and for $t=0$

$$|u| \lesssim (1+r)^{\gamma+\beta}.$$

Then we have the following estimates

$$|u| \lesssim (1+s)^\gamma \max(1,(1+|q|)^{\alpha+1}), \ for\ q<0, \quad |u| \lesssim (1+s)^\gamma (1+|q|)^{\beta+1} \ for\ q>0.$$

Proof. We assume first $q>0$. We integrate the estimate

$$|\partial_q u| \lesssim (1+s)^\gamma (1+|q|)^\beta,$$

from $t=0$. We obtain, since $\beta < -1$, for $q>0$

$$|u| \lesssim (1+s)^\gamma (1+|q|)^{\beta+1}.$$

Consequently, we have, for $q=0$, $|u| \lesssim (1+s)^\gamma$. We now assume $q<0$. We integrate

$$|\partial_q u| \lesssim (1+s)^\gamma (1+|q|)^\alpha,$$

from $q=0$. We obtain

$$|u| \lesssim (1+s)^\gamma \max(1,(1+|q|)^{\alpha+1}).$$

This concludes the proof of Lemma 3.1.11. □

3.1.6 Main Result

We introduce an other cut-off function $\Upsilon : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\Upsilon(\rho)=0$ for $\rho \leq \frac{1}{2}$ and $\rho \geq 2$ and $\Upsilon = 1$ for $\frac{3}{4} \leq \rho \leq \frac{3}{2}$. Theorem 3.1.12 is our main result, in which we prove stability of Minkowski space-time with a translational symmetry in exponential time $T \lesssim \exp\left(\frac{1}{\sqrt{\varepsilon}}\right)$ where $\varepsilon > 0$ is the size of the small initial data.

Theorem 3.1.12. *Let $0<\varepsilon<1$. Let $\frac{1}{2} < \delta < 1$ and $N \geq 40$. Let $(\varphi_0, \varphi_1) \in H^{N+1}_\delta(\mathbb{R}^2) \times H^N_{\delta+1}(\mathbb{R}^2)$. We assume*

$$\|\varphi_0\|_{H^{N+1}_\delta} + \|\varphi_1\|_{H^N_{\delta+1}} \leq \varepsilon.$$

Let $T \lesssim \exp(\frac{1}{\sqrt{\varepsilon}})$. Let $0 < \rho \ll \sigma \ll \mu \ll \delta$. If ε is small enough, there exists $b(\theta), J(\theta) \in W^{N,2}(\mathbb{S}^1)$ and there exists a global coordinate chart (t, x_1, x_2) such that, for $t \leq T$, there exists a solution (φ, g) of (3.1.1) that we can write

$$g = g_b + \Upsilon\left(\frac{r}{t}\right)\left(\frac{g_{\underline{L}\underline{L}}}{4}dq^2 + \frac{g_{U\underline{L}}}{2} r dq d\theta\right) + \tilde{g}$$

such that we have the estimates

$$\sum_{|I|\leq N}\Bigg(\|\alpha_2 w_0^{\frac{1}{2}}(q)\partial Z^I\varphi\|_{L^2} + \frac{1}{\sqrt{1+t}}\|\alpha_2 w_3^{\frac{1}{2}}(q)\partial Z^I g_{\underline{L}\underline{L}}\|_{L^2} + \frac{1}{\sqrt{1+t}}\|\alpha_2 w_3^{\frac{1}{2}}(q)\partial Z^I g_{\underline{L}U}\|_{L^2}$$
$$+ \|\alpha_2 w_2^{\frac{1}{2}}(q)\partial Z^I \widetilde{g}\|_{L^2}\Bigg) \lesssim \varepsilon(1+t)^{C\sqrt{\varepsilon}}.$$

with

$$\begin{cases} w_0(q) = (1+|q|)^{2+2\delta}, \ q>0 \\ w_0(q) = 1 + \frac{1}{(1+|q|)^{2\mu}}, \ q<0, \end{cases}$$

$$\begin{cases} w_2(q) = (1+|q|)^{2+2\delta}, \ q > 0 \\ w_2(q) = \frac{1}{(1+|q|)^{1+2\mu}}, \ q < 0, \end{cases}$$

$$\begin{cases} w_3(q) = (1+|q|)^{3+2\delta}, \ q > 0 \\ w_3(q) = 1 + \frac{1}{(1+|q|)^{2\mu}}, \ q < 0, \end{cases}$$

$$\begin{cases} \alpha_2(q) = (1+|q|)^{-2\sigma}, \ q > 0 \\ \alpha_2(q) = 1, \ q < 0, \end{cases}$$

Moreover, for all $\rho > 0$, we have the L^∞ estimate, for $|I| \leq \frac{N}{2} + 2$ and $r < t$

$$\begin{aligned} |Z^I \varphi(x,t)| &\lesssim \frac{\varepsilon C(\rho)}{(1+t+r)^{\frac{1}{2}}(1+|t-r|)^{\frac{1}{2}-4\rho}}, \\ |Z^I g_{\underline{LL}}| &\lesssim \frac{\varepsilon C(\rho)}{(1+|t-r|)^{\frac{1}{2}-\rho}}, \\ |Z^I g_{\underline{L}U}| + |Z^I \widetilde{g}| &\lesssim \frac{\varepsilon C(\rho)}{(1+t+r)^{\frac{1}{2}-\rho}}. \end{aligned}$$

and we have the estimate for b

$$\left| b(\theta) + \int_{\Sigma_{T,\theta}} (\partial_q \varphi)^2 r dr \right| \lesssim \frac{\varepsilon^2}{\sqrt{T}},$$

where we have used the notation

$$\int_{\Sigma_{T,\theta}} (\partial_q \varphi)^2 r dr = \int_0^\infty \left(\partial_q \varphi(T,r,\theta)\right)^2 r dr. \tag{3.1.15}$$

Comments on Theorem 3.1.12

- We consider perturbations of $3+1$ dimensional Minkowski space-time with a translational space-like Killing field. These perturbations are not asymptotically flat in $3+1$ dimensions, therefore the result of Theorem 3.1.12 does not follow from the stability of Minkowski space-time by Christodoulou and Klainerman [13].

- As our gauge, we choose the generalized wave coordinates, which are picked such that the generalized wave coordinates condition is satisfied by g_b. Therefore, the method we use has a lot in common with the method of Lindblad and Rodnianski in [40] where they proved the stability of Minkowski space-time in harmonic gauge. It is an interesting problem to investigate the stability of Minkowski with a translation symmetry using a strategy in the spirit of [13] or [34].

- The function $J(\theta)$, and the quantities

$$\int b(\theta) d\theta, \quad \int b(\theta)\cos(\theta) d\theta, \quad \int b(\theta)\sin(\theta) d\theta$$

  are imposed by the constraint equations for the initial data (see Theorem 3.1.3). The quantity $\int b(\theta)d\theta$ is called angle, and the vector $(\int b(\theta)\cos(\theta)d\theta, \int b(\theta)\sin(\theta)d\theta)$ is called linear momentum. We can make a rapprochement of these quantities with the ADM mass and linear momentum. The remaining Fourier coefficients of b are chosen to ensure the convergence to Minkowski in the direction of time-like infinity, and is an essential element in the proof of the quasi stability.

- The logarithmic growth of $\|w^{\frac{1}{2}}(q)\partial Z^N \varphi\|_{L^2}$, and the condition
$$b(\theta) \sim \int_{\Sigma_{T,\theta}} (\partial_q \varphi)^2 \, r dr, \tag{3.1.16}$$
give the estimate $|\partial^N b| \lesssim \varepsilon^2 (1+T)^{C\varepsilon}$. To avoid factors of the form $(1+T)^{C\varepsilon}$ in all our estimate, we are forced to assume $(1+T)^{C\varepsilon} \lesssim 1$. This is the only place where we need $(1+T)^{C\varepsilon} \lesssim 1$, and this is what prevents us to prove the stability.
- The condition (3.1.16) is not necessary to control the metric in the exterior region $r > t$. For this reason we believe that the stability holds in the exterir region, without the condition $T \lesssim \exp\left(\frac{1}{\sqrt{\varepsilon}}\right)$.

As we said in the second comment, we use a method similar than Lindblad and Rodnianski method in [40]. Let us list some of the similarities and differences with their method.

Similarities with [40]

- We use the vector field method. The vector fields we use are Klainerman vector fields of Minkowski space-time.
- We use the wave coordinate condition to obtain more decay on the coefficients $\widetilde{g}_{\mathcal{T}\mathcal{T}}$ of the metric.
- We exhibit the structure corresponding to the model problem (3.1.3).

Differences with [40]

- The asymptotic behaviour given by the solutions of the constraint equations prevent us to work in wave coordinates. Instead we work in generalised wave coordinates.
- In the exterior region, our solution do not converge to Minkowski, but to a family of Ricci flat metrics g_b.
- The decay of the free wave is weaker in $2+1$ dimension. Consequently, the coefficient $g_{\underline{LL}}$ of the metric does not have any decay near the light cone. We have to rely on the null decomposition at all steps in our proof to isolate this behaviour, even in the L^2 estimates.
- We have to fit $b(\theta)$ so that the condition (3.1.16) is satisfied. This lead to regularity issues for b, which prevent us from proving the global existence.

The structure of the paper is as followed. In Section 3.2 we describe the structure of the equations (3.1.1) in generalized wave coordinates. We exhibit the structure of our system in Section 3.2. We also describe the interactions between g_b and $\widetilde{g}$. In Section 3.1.3 we outline the main issues of the proof by discussing some model problems. In section 3.4 we give our bootstrap assumption. In section 3.5 we derive preliminaries estimates thanks to the wave coordinate condition. In section 3.6 we derive preliminaries estimate for the angle and the linear momentum. In section 3.7, we will exploit the analysis begun in section 3.1.2. In section 3.8.4 we will improve the L^∞ estimate. In section 3.9 we will derive the weighted energy estimate. In section 3.10 we will improve the L^2 estimates and in section 3.11 we will adjust the parameter $b(\theta)$.

3.2 Structure of the equations

In this section, we provide a discussion of the specific features of the structure of the equations, which will be relevant for the proof of Theorem 3.1.12.

3.2.1 The generalized wave coordinates

Wave coordinates allow to recast Einstein equations as a system of non-linear wave equations. The wave coordinates condition, which consists in choosing coordinates such that $\Box_g x^\alpha = 0$ can be rewritten as

$$g^{\lambda\beta}\Gamma^\alpha_{\lambda\beta} = 0.$$

However, for the metric g_b defined by (3.1.6), the coordinates (t, x_1, x_2) are not wave coordinates, not even asymptotically. We will therefore work with generalized wave coordinates. We will impose that our metric satisfies

$$g^{\lambda\beta}\Gamma^\alpha_{\lambda\beta} = H^\alpha_b$$

where H^α_b is defined by (3.1.9)

$$H^\alpha_b = (g_b)^{\lambda\beta}(\Gamma_b)^\alpha_{\lambda\beta} + F^\alpha,$$

with F^α of the form

$$\widetilde{g}\frac{q\chi(q)\partial_\theta b}{r^2}.$$

The role of F^α was explained in section 3.1.4. In generalized wave coordinates, the expression (3.12.11) of Appendix 3.12.2 allow us to write the system (3.1.1) under the form

$$\begin{cases} \Box_g \varphi = 0 \\ \Box_g g_{\mu\nu} = -2\partial_\mu\varphi\partial_\nu\varphi + P_{\mu\nu}(\partial g, \partial g) + g_{\mu\rho}\partial_\nu H^\rho + g_{\nu\rho}\partial_\mu H^\rho, \end{cases} \tag{3.2.1}$$

where

$$\begin{aligned} P_{\mu\nu}(g)(\partial g, \partial g) =& \frac{1}{2}g^{\alpha\rho}g^{\beta\sigma}\left(\partial_\mu g_{\rho\sigma}\partial_\alpha g_{\beta\nu} + \partial_\nu g_{\rho\sigma}\partial_\alpha g_{\beta\mu} - \partial_\beta g_{\mu\rho}\partial_\alpha g_{\nu\sigma} - \frac{1}{2}\partial_\mu g_{\alpha\beta}\partial_\nu g_{\rho\sigma}\right) \\ &+ \frac{1}{2}g^{\alpha\beta}g^{\lambda\rho}\partial_\alpha g_{\nu\rho}\partial_\beta g_{\mu\rho}. \end{aligned} \tag{3.2.2}$$

Remark 3.2.1. *In generalized wave coordinates, the wave operator can be expressed as*

$$\Box_g = g^{\alpha\rho}\partial_\alpha\partial_\rho - H^\rho_b\partial_\rho.$$

The expression (3.12.11) yields also

$$(R_b)_{\mu\nu} = -\frac{1}{2}\Box_{g_b}(g_b)_{\mu\nu} + \frac{1}{2}P_{\mu\nu}(g_b)(\partial g_b, \partial g_b) + \frac{1}{2}\left((g_b)_{\mu\rho}\partial_\nu \bar{H}^\rho_b + (g_b)_{\mu\rho}\partial_\mu \bar{H}^\rho_b\right). \tag{3.2.3}$$

Therefore, subtracting twice the equation (3.2.3) to the second equation of (3.2.1) we obtain

$$\begin{cases} \Box_g \varphi = 0, \\ \Box_g \widetilde{g}_{\mu\nu} = -2\partial_\mu\varphi\partial_\nu\varphi + 2(R_b)_{\mu\nu} + P_{\mu\nu}(g)(\partial\widetilde{g}, \partial\widetilde{g}) + \widetilde{P}_{\mu\nu}(\widetilde{g}, g_b), \end{cases} \tag{3.2.4}$$

where $P_{\mu\nu}(g)(\partial\widetilde{g}, \partial\widetilde{g})$ is defined by (3.2.2) and

$$\begin{aligned} \widetilde{P}_{\mu\nu}(\widetilde{g}, g_b) = & \left(g_b^{\alpha\beta} - g^{\alpha\beta}\right)\partial_\alpha\partial_\beta(g_b)_{\mu\nu} + F^\rho\partial_\rho(g_b)_{\mu\nu} \\ &+ P_{\mu\nu}(g)(\partial g, \partial_g) - P_{\mu\nu}(g)(\partial\widetilde{g}, \partial\widetilde{g}) - P_{\mu\nu}(g_b)(\partial g_b, \partial g_b) \\ &+ (g_b)_{\mu\rho}\partial_\nu F^\rho + (g_b)_{\nu\rho}\partial_\mu F^\rho + \widetilde{g}_{\mu\rho}\partial_\nu H^\rho_b + \widetilde{g}_{\nu\rho}\partial_\mu H^\rho_b. \end{aligned} \tag{3.2.5}$$

Let us note that $\widetilde{P}_{\mu\nu}(\widetilde{g}, g_b)$ contains only crossed terms between g_b and $\widetilde{g}$.

3.2.2 The weak null structure

To exhibit the main terms in the structure of (3.2.4), let us neglect for a moment $P_{\mu\nu}$, $\widetilde{P}_{\mu\nu}$, H_b. We will see in the next section that this approximation is relevant. Let us also neglect the nonlinear terms involving $\bar{\partial}$ derivatives. Then we obtain the following approximate system

$$\begin{aligned}
\Box\varphi + g_{LL}\partial_q^2\varphi &= 0,\\
\Box g_{\mathcal{T}\mathcal{V}} + g_{LL}\partial_q^2 g_{\mathcal{T}\mathcal{V}} &= 0,\\
\Box g_{\underline{LL}} + g_{LL}\partial_q^2 g_{\underline{LL}} &= 4\left(-2(\partial_q\varphi)^2 - 2b(\theta)\frac{\partial_q^2(\chi(q)q)}{r}\right),
\end{aligned}$$

where we also have used the approximation

$$(R_b)_{qq} \sim -\frac{b(\theta)\partial_q^2(q\chi(q))}{r} + O\left(\frac{C(b,b',J,J')\mathbb{1}_{1<q<2}}{r^2}\right),$$

as shown in (3.1.7). In $2+1$ dimensions, a term of the form $g_{LL}\partial_q^2\varphi$ is impossible to handle if one only relies on the decay for g_{LL} provided by the fact of being a solution of a wave equation. However, as in [40], we can exploit the wave condition to obtain better decay for some coefficients of the metric. More precisely, we have roughly

$$\partial g_{\mathcal{T}\mathcal{T}} \sim \bar{\partial} g.$$

This is done properly in Proposition 3.5.1 for the coefficient g_{LL} and in Proposition 3.5.2 for the coefficients g_{LU} and g_{UU}. Therefore, the $g_{\mathcal{T}\mathcal{T}}$ coefficients have a better decay in t than the solutions of the wave equation (the challenges of the quasilinear terms of the form $g_{LL}\partial_q^2\varphi$, $g_{LL}\partial_q^2 g_{\mathcal{T}\mathcal{V}}$ are presented in Section 3.3.4).

Remark 3.2.2. *The other quasilinear terms are of the form*

$$g_{\mathcal{T}\mathcal{V}}\partial_{\mathcal{T}}\partial_{\mathcal{V}}\varphi, \quad g_{\mathcal{T}\mathcal{V}}\partial_{\mathcal{T}}\partial_{\mathcal{V}}\widetilde{g}.$$

Consequently, they involved at least one "good derivative" of $\varphi, \widetilde{g}$. Thus, they are easier to estimate, and we can always focus on the terms

$$g_{LL}\partial_q^2\varphi, \quad g_{LL}\partial_q^2\widetilde{g}.$$

Assuming that we can also neglect the terms involving g_{LL}, we are reduced to the following system

$$\begin{cases}
\Box\varphi = 0,\\
\Box g_{\underline{LL}} = 4\left(-2(\partial_q\varphi)^2 - 2b(\theta)\frac{\partial_q^2(\chi(q)q)}{r}\right),
\end{cases} \tag{3.2.6}$$

which is a system of the form (3.1.3) and displays the weak null structure.

The second component of the solution of (3.1.3) do not have any decay near the light cone in $2+1$ dimensions (see Section 3.1.2 for the radial case). Therefore, the coefficient $g_{\underline{LL}}$ will not display any decay at all near the light cone (see the estimates of Theorem 3.1.12). To obtain decay for $g_{\underline{LL}}$ in the q variable, we will approximate $\frac{g_{\underline{LL}}}{4}$ by the solution h_0 of the following transport equation

$$\partial_q h_0 = -2r(\partial_q\varphi)^2 - 2b(\theta)\partial_q^2(q\chi(q)).$$

The ideas of this approximation are presented in Section 3.3.2, and are exploited in Section 3.7.

3.2.3 Non-commutation of the wave operator with the null frame

The structure of Einstein equations can only be seen in the null frame. However it is well known that the wave operator does not commute with the null frame. In Theorem 3.1.12 we have decomposed our metric in the following way

$$g = g_b + \widetilde{g} + \Upsilon\left(\frac{r}{t}\right)\left(\frac{g_{\underline{L}\underline{L}}}{4}dq^2 + \frac{g_{U\underline{L}}}{2}rdqd\theta\right).$$

The problems of non-commutation induced by $g_{\underline{L}\underline{L}}$ and $g_{U\underline{L}}$ are totally similar. Consequently, we can neglect the second one. We expressed the 2-forms dq^2 in the coordinate (t, x_1, x_2)

$$dq^2 = (dr - dt)^2 = (\cos(\theta)dx^1 + \sin(\theta)dx^2 - dt)^2$$

Therefore, we will have, in the coordinates x_1, x_2

$$\Box\left(\Upsilon\left(\frac{r}{t}\right)g_{\underline{L}\underline{L}}dq^2\right)_{\mu\nu} - \Box\left(\Upsilon\left(\frac{r}{t}\right)g_{\underline{L}\underline{L}}\right)(dq^2)_{\mu\nu} = \Upsilon\left(\frac{r}{t}\right)\frac{1}{r^2}\left(u^1_{\mu\nu}(\theta)g_{\underline{L}\underline{L}} + u^2_{\mu\nu}(\theta)\partial_\theta g_{\underline{L}\underline{L}}\right) \tag{3.2.7}$$

where $u^1_{\mu\nu}$ and $u^2_{\mu\nu}$ are some trigonometric functions. The challenges of the terms involving $u^1_{\mu\nu}$ and $u^2_{\mu\nu}$ are explained in Section 3.3.3.

3.2.4 The semi linear term $P_{\mu\nu}(g)(\partial\widetilde{g}, \partial\widetilde{g})$.

Recall the form of the term $P_{\mu\nu}(g)(\partial\widetilde{g}, \partial\widetilde{g})$.

$$\begin{aligned} P_{\mu\nu}(g)(\partial\widetilde{g}, \partial\widetilde{g}) =& \frac{1}{2}g^{\alpha\rho}g^{\beta\sigma}\left(\partial_\mu\widetilde{g}_{\rho\sigma}\partial_\alpha\widetilde{g}_{\beta\nu} + \partial_\nu\widetilde{g}_{\rho\sigma}\partial_\alpha\widetilde{g}_{\beta\mu} - \partial_\beta\widetilde{g}_{\mu\rho}\partial_\alpha\widetilde{g}_{\nu\sigma} - \frac{1}{2}\partial_\mu\widetilde{g}_{\alpha\beta}\partial_\nu\widetilde{g}_{\rho\sigma}\right) \\ &+ \frac{1}{2}g^{\alpha\beta}g^{\lambda\rho}\partial_\alpha\widetilde{g}_{\nu\rho}\partial_\beta\widetilde{g}_{\mu\rho}. \end{aligned}$$

The quadratic terms In the null frame $(L, \underline{L}, U)$ the only non zero coefficients of the Minkowski metric are $m^{L\underline{L}} = -\frac{1}{2}$ and $m^{UU} = 1$. Thanks to this remark, we can describe the terms appearing in the different components of $P_{\mu\nu}$.

- In $P_{\mathcal{T}\mathcal{T}}(g)(\partial\widetilde{g}, \partial\widetilde{g})$, there can not be strictly more than 2 occurrences of the vector field $\underline{L}$. Therefore, the quadratic terms are of one of these form

$$\partial_{\mathcal{V}}\widetilde{g}_{\mathcal{V}\mathcal{T}}\partial_{\mathcal{T}}\widetilde{g}_{\mathcal{T}\mathcal{T}}, \partial_{\mathcal{T}}\widetilde{g}_{\mathcal{V}\mathcal{V}}\partial_{\mathcal{T}}\widetilde{g}_{\mathcal{T}\mathcal{T}}, \tag{3.2.8}$$

  where we have used the fact, proved in Section 3.5 that

$$\partial_{\mathcal{V}}\widetilde{g}_{\mathcal{T}\mathcal{T}} \sim \partial_{\mathcal{T}}\widetilde{g}_{\mathcal{V}\mathcal{V}}.$$

  These terms all have the classical null structure. How this structure can be used to show global existence is explained in Section 3.3.1. Since they are by far easier to handle than the one we will describe in the following, they will be neglected in the proof of Theorem 3.1.12.

- In $P_{\mathcal{T}\mathcal{V}}(g)(\partial\widetilde{g}, \partial\widetilde{g})$, there can not be strictly more than 3 occurrences of the vector field $\underline{L}$. Therefore, the quadratic terms are of one of these form

$$\partial_{\mathcal{V}}\widetilde{g}_{\mathcal{T}\mathcal{V}}\partial_{\mathcal{T}}\widetilde{g}_{\mathcal{T}\mathcal{V}}, \quad \partial_{\mathcal{V}}\widetilde{g}_{\mathcal{V}\mathcal{V}}\partial_{\mathcal{T}}\widetilde{g}_{\mathcal{T}\mathcal{T}}, \quad \partial_{\mathcal{T}}\widetilde{g}_{\mathcal{V}\mathcal{V}}\partial_{\mathcal{V}}\widetilde{g}_{\mathcal{T}\mathcal{T}}, \quad \partial_{\mathcal{T}}\widetilde{g}_{\mathcal{T}\mathcal{V}}\partial_{\mathcal{T}}\widetilde{g}_{\mathcal{V}\mathcal{V}}$$

  where we have used the fact, proved in Section 3.5 that

$$\partial_{\mathcal{V}}\widetilde{g}_{\mathcal{T}\mathcal{T}} \sim \partial_{\mathcal{T}}\widetilde{g}_{\mathcal{V}\mathcal{V}}.$$

These terms all have the null structure. However, since $g_{\underline{L}\underline{L}}$ does not decay at all in t (see the estimates of Theorem 3.1.12), one has to be more careful with the terms of the form

$$\partial_{\mathcal{T}} g_{\mathcal{T}\mathcal{T}} \partial_{\underline{L}} g_{\underline{L}\underline{L}}$$

These terms have a good structure since $\partial_{\mathcal{T}} g_{\mathcal{T}\mathcal{T}}$ is a "good derivative" of a "good component". However, one needs two steps to exploit this structure, which can be difficult to achieve if there is no regularity left. Thankfully, these terms have three occurrences of $\underline{L}$, therefore they can only intervene in $P_{\mathcal{T}\underline{L}}$.

- In $P_{\underline{L}L}$ we will have to be careful with

$$\partial_L \widetilde{g}_{LL} \partial_{\underline{L}} \widetilde{g}_{\underline{L}\underline{L}}.$$

This term can be converted in $\partial_{\underline{L}} \widetilde{g}_{LL} \partial_L \widetilde{g}_{\underline{L}\underline{L}}$ with the help of the algebraic trick

$$\Box(uv) = u\Box v + v\Box u + \partial_L u \partial_{\underline{L}} v + \partial_L v \partial_{\underline{L}} u + \partial_U u \partial_U v.$$

This fact will be used only in the proof of Lemma 3.10.6.

- In $P_{\underline{L}U}$ we will have to be careful with

$$\partial_U g_{LL} \partial_{\underline{L}} g_{\underline{L}\underline{L}}.$$

This term can not be removed with the previous trick. We will have to single out its influence thanks to the decomposition

$$g = g_b + \chi\left(\frac{r}{t}\right) h dq^2 + \chi\left(\frac{r}{t}\right) k r dq d\theta + \widetilde{g}_4,$$

where k satisfies

$$\Box_g k = \partial_U \widetilde{g}_{LL} \partial_{\underline{L}} \widetilde{g}_{\underline{L}\underline{L}}.$$

This will also be used only in the proof of Lemma 3.10.6.

- The terms in $P_{\underline{L}\underline{L}}$ which are not of the previous form can be written

$$\partial_{\underline{L}} g_{LL} \partial_{\underline{L}} g_{\underline{L}\underline{L}}, \quad \partial_{\underline{L}} g_{L\underline{L}} \partial_L g_{\underline{L}\underline{L}}. \tag{3.2.9}$$

We note the crucial cancellation of terms of the form $(\partial_{\underline{L}} g_{\underline{L}\underline{L}})^2$ in $P_{\underline{L}\underline{L}}$. The contributions (3.2.9) will be single out in (3.2.12).

The cubic terms In two dimensions, cubic terms could be troublesome. However, in the form $P_{\mathcal{V}\mathcal{T}}$, if there are 4 occurrences of the vector field $\underline{L}$, or in $P_{\underline{L}\underline{L}}$ if there are 5 occurrences of the vector field $\underline{L}$, then we have a factor $g^{\underline{L}\underline{L}}$, which has a decay equivalent to g_{LL}. Therefore we can neglect the cubic terms in this nonlinearity.

3.2.5 The crossed terms

In this section, we discuss the structure of the crossed terms between b and $(\widetilde{g}, \varphi)$.

The crossed terms involving two derivatives of b are absent In the expression

$$\Box_g g_{\mu\nu} - (g_{\mu\rho}\partial_\nu H_b^\rho + g_{\nu\rho}\partial_\mu H_b^\rho),$$

there could be terms involving two derivatives of $b(\theta)$, which would be troublesome since they would lead to a loss of a derivative (recall that we only have the regularity $b \in W^{N,2}$). However, the terms involving two derivatives of b in this expression, are the same than the terms involving two derivatives of b in $R_{\mu\nu}(g)$. Thus, these terms cancel in the expression

$$\left(g_b^{\alpha\beta} - g^{\alpha\beta}\right)\partial_\alpha\partial_\beta(g_b)_{\mu\nu} + (g_b)_{\mu\rho}\partial_\nu F^\rho + (g_b)_{\nu\rho}\partial_\mu F^\rho + \widetilde{g}_{\mu\rho}\partial_\nu H_b^\rho + \widetilde{g}_{\nu\rho}\partial_\mu H_b^\rho,$$

which appears in $\widetilde{P}_{\mu\nu}(\widetilde{g}, g_b)$ defined by (3.2.5). These cancellations can be checked for example with Mathematica.

The crossed terms in $\widetilde{P}_{\mu\nu}$ We recall from (3.1.6) that

$$g_b = dsdq + (r + \chi(q)qb(\theta))^2 d\theta^2 + J(\theta)\chi(q)dqd\theta.$$

Therefore in $\widetilde{P}_{\mu\nu}$ we can find terms involving

$$(g_b)_{UU} = \left(1 + \frac{\chi(q)qb(\theta)}{r}\right)^2 \quad and \quad (g_b)_{U\underline{L}} = -2\frac{J(\theta)\chi(q)}{r},$$

Since $(g_b)_{U\underline{L}}$ decays faster than $(g_b)_{UU}$ let us focus on the crossed terms between $(g_b)_{UU}$ and $\widetilde{g}$. The problem with the term $(g_b)_{UU}$ is that far from the light cone, it does not decay at all. This is one of the causes of the logarithmic growth of the energy in the statement of Theorem 3.1.12. However, these terms are present only in the exterior region. Moreover they display also a special structure. Since the terms involving two derivatives of b are absent, and the terms involving two derivatives of $\widetilde{g}$ are only present in $\Box_g\widetilde{g}$, the terms in $\widetilde{P}_{\mu\nu}$ are of the form

$$g^{--}\partial_-(g_b)_{UU}\partial_- g_{--}.$$

- In $\widetilde{P}_{\mathcal{T}\mathcal{V}}$ the crossed terms involving $\partial_{\underline{L}}(g_b)_{UU}$ can not contain more than two occurrences of $\underline{L}$. They must be of the following form

$$\partial_{\underline{L}}(g_b)_{UU}\partial_{\mathcal{T}}\widetilde{g}_{\mathcal{T}\mathcal{V}}, \quad \partial_{\mathcal{T}}(g_b)_{UU}\partial_{\mathcal{V}}\widetilde{g}_{\mathcal{T}\mathcal{V}}, \quad \partial_{\mathcal{T}}(g_b)_{UU}\partial_{\mathcal{T}}\widetilde{g}_{\mathcal{V}\mathcal{V}},$$

  where we have used the wave coordinates condition $\partial_{\mathcal{V}}\widetilde{g}_{\mathcal{T}\mathcal{T}} \sim \partial_{\mathcal{T}}\widetilde{g}_{\mathcal{T}\mathcal{V}}$. We have the following inequalities, thanks to (3.1.14)

$$|\partial_{\underline{L}}(g_b)_{UU}\partial_{\mathcal{T}}\widetilde{g}_{\mathcal{T}\mathcal{V}}| \lesssim \frac{\mathbb{1}_{q>0}(|b| + |\partial_\theta b|)}{1+r}|\partial_{\mathcal{T}}\widetilde{g}_{\mathcal{T}\mathcal{V}}| \lesssim \frac{\mathbb{1}_{q>0}(|b| + |\partial_\theta b|)}{(1+r)^2}|Z^1\widetilde{g}_{\mathcal{T}\mathcal{V}}|,$$

$$|\partial_{\mathcal{T}}(g_b)_{UU}\partial_{\mathcal{V}}\widetilde{g}_{\mathcal{T}\mathcal{V}}| \lesssim \frac{\mathbb{1}_{q>0}(1+|q|)(|b| + |\partial_\theta b|)}{(1+r)^2}|\partial_{\mathcal{V}}\widetilde{g}_{\mathcal{T}\mathcal{V}}| \lesssim \frac{\mathbb{1}_{q>0}(|b| + |\partial_\theta b|)}{(1+r)^2}|Z^1\widetilde{g}_{\mathcal{T}\mathcal{V}}|.$$

  These two contributions are therefore quite similar. In the following, it will be sufficient to study the term

$$\partial_{\underline{L}}(g_b)_{UU}\partial_{\mathcal{T}}\widetilde{g}_{\mathcal{T}\mathcal{V}}. \tag{3.2.10}$$

  The challenges of this terms will be discussed in Section 3.3.5

- In $\widetilde{P}_{\underline{L}\underline{L}}$, we may have three occurrences of $\underline{L}$. Therefore there are terms of the form

$$\partial_{\underline{L}}(g_b)_{UU}\partial_{\mathcal{T}} g_{\underline{L}\underline{L}}, \quad \partial_{\underline{L}}(g_b)_{UU}\partial_{\underline{L}} g_{\underline{L}\underline{L}}, \quad \partial_{\mathcal{T}} g_{UU}\partial_{\underline{L}} g_{\underline{L}\underline{L}}.$$

  We have the following inequalities, thanks to (3.1.14)

$$|\partial_{\underline{L}}(g_b)_{UU}\partial_{\mathcal{T}} g_{\underline{L}\underline{L}}| \lesssim \frac{\mathbb{1}_{q>0}(|b| + |\partial_\theta b|)}{1+r}|\partial_{\mathcal{T}}\widetilde{g}_{\underline{L}\underline{L}}| \lesssim \frac{\mathbb{1}_{q>0}(|b| + |\partial_\theta b|)}{(1+r)^2}|Z^1\widetilde{g}_{\underline{L}\underline{L}}|$$

$$|\partial_{\underline{L}}(g_b)_{UU}\partial_{\underline{L}} g_{\underline{L}\underline{L}}| \lesssim \frac{\mathbb{1}_{q>0}(|b| + |\partial_\theta b|)}{1+r}|\partial_{\underline{L}}\widetilde{g}_{\underline{L}}| \lesssim \frac{\mathbb{1}_{q>0}(|b| + |\partial_\theta b|)}{(1+r)(1+|q|)}|Z^1\widetilde{g}_{L\underline{L}}|$$

$$|\partial_{\mathcal{T}}(g_b)_{UU}\partial_{\underline{L}} g_{\underline{L}\underline{L}}| \lesssim \frac{\mathbb{1}_{q>0}(1+|q|)(|b| + |\partial_\theta b|)}{(1+r)^2}|\partial_{\underline{L}}\widetilde{g}_{\underline{L}\underline{L}}| \lesssim \frac{\mathbb{1}_{q>0}(|b| + |\partial_\theta b|)}{(1+r)^2}|Z^1\widetilde{g}_{\underline{L}\underline{L}}|.$$

Consequently, the worst term is

$$\partial_{\underline{L}}(g_b)_{UU}\partial_{\underline{L}}g_{\underline{L}L}. \tag{3.2.11}$$

We introduce the following notation, to single out the contributions of (3.2.11) and (3.2.9)

$$Q_{\underline{L}\underline{L}}(h,\widetilde{g}) = \partial_{\underline{L}}g_{LL}\partial_{\underline{L}}h + \partial_{\underline{L}}g_{L\underline{L}}\partial_L h + \partial_{\underline{L}}(g_b)_{UU}\partial_{\underline{L}}g_{\underline{L}L}. \tag{3.2.12}$$

The crossed terms involving two derivatives of $\widetilde{g}$ With our choice of coordinates, these terms only appear in $\Box_g\widetilde{g}$. They are of the form

$$\mathbb{1}_{q>0}\frac{b(1+|q|)}{1+r}\partial_U^2\widetilde{g}.$$

Their contribution is most of the time similar than the one of (3.2.10), except in the energy estimate, where they require a special treatment because of their lack of decay far from the light cone (see Section 3.9).

The crossed terms in $\Box_g\varphi$ The crossed terms between g_b and $\partial\varphi$ are of the form

$$g^{--}\partial_-(g_b)_{UU}\partial_-\varphi.$$

Consequently, they must be of the following form

$$\partial_{\mathcal{V}}(g_b)_{UU}\partial_{\mathcal{T}}\varphi, \quad \partial_{\mathcal{T}}(g_b)_{UU}\partial_{\mathcal{V}}\varphi.$$

Like for $\widetilde{P}_{\mathcal{V}\mathcal{T}}$, it will be sufficient to study

$$\partial_{\mathcal{V}}(g_b)_{UU}\partial_{\mathcal{T}}\varphi. \tag{3.2.13}$$

The crossed terms between g_b and $\partial^2\varphi$ are of the form

$$\mathbb{1}_{q>0}\frac{b(1+|q|)}{1+r}\partial_U^2\widetilde{\varphi}.$$

As for $\widetilde{g}$, their contribution is most of the time similar than the one of (3.2.13), except in the energy estimate, where they require a special treatment because of their lack of decay far from the light cone (see Section 3.9).

Remark 3.2.3. *In the region $q>0$ it is generally sufficient to study the crossed terms. Indeed, the crossed terms are the one presenting the less decay far from light cone.*

3.3 Model problems

The proof relies on a bootstrap scheme. Roughly speaking, we will assume some estimates on the coefficients $Z^I\varphi, Z^I g_{\underline{L}\underline{L}}$ and $Z^I g_{\mathcal{T}\mathcal{V}}$:

- L^∞ estimates for $I \le \frac{N}{2}$,
- L^2 estimates for $I \le N$.

We rewrite the bootstrap assumptions in the condensed form

$$|\varphi|_{X_1} \le 2C_0\varepsilon, \quad |g|_{X_2} \le 2C_0\varepsilon,$$

where C_0 is a constant depending only on the quantities ρ,σ,μ,δ,N introduced in the statement of Theorem 3.1.12 and such that at $t=0$

$$|\varphi|_{X_1} \le C_0\varepsilon, \quad |g|_{X_2} \le C_0\varepsilon.$$

Thanks to the $L^\infty - L^\infty$ estimate and the energy estimate, we will be able to prove

$$|\varphi|_{X_1} \leq C_0\varepsilon + C\varepsilon^2, \quad |g|_{X_2} \leq C_0\varepsilon + C\varepsilon^2.$$

Therefore, for ε chosen small enough so that $C\varepsilon \leq \frac{C_0}{2}$, this improves the bootstrap assumptions.

We will first consider a toy model, which exhibits some of the mechanisms involved in the proof.

3.3.1 Global well posedness for a semi linear wave equation with the null structure

We consider the following $2+1$ dimensional semi-linear wave equation

$$\begin{cases} \Box u = \partial u \bar{\partial} u, \\ (u, \partial_t u)|_{t=0} = (u_0, u_1). \end{cases} \tag{3.3.1}$$

Note that the nonlinearity satisfies the null condition. Consequently, this model will show us how to treat the terms of the form (3.2.8). The following result is proved in [28]. We will give a proof of it for sake of completeness, and because it exhibits some of the mechanisms involved in the proof of Theorem 3.1.12.

Proposition 3.3.1. *Let $0 < \delta < \frac{1}{2}$. Let $N \geq 8$. Let $u_0, u_1 \in \times H^{N+1}_{-\frac{1}{2}+\delta} \times H^N_{\delta+\frac{1}{2}}$ such that*

$$\|u_0\|_{H^{N+1}_{-\frac{1}{2}+\delta}} + \|u_1\|_{H^N_{\delta+\frac{1}{2}}} \leq \varepsilon.$$

If $\varepsilon > 0$ is small enough, the equation (3.3.1) *has a global solution u.*

Proof. Let $0 < \mu < \frac{1}{4}$. We introduce the weight function

$$\begin{cases} w(q) = 1 + \frac{1}{(1+|q|)^{2\mu}}, \ q < 0, \\ w(q) = (1+|q|)^{1+2\delta} \ q > 0. \end{cases}$$

Let $0 < \rho < \frac{\delta}{2}$. To prove global existence for equation (3.3.1), we consider a time $T > 0$ such that, on $0 \leq t \leq T$

$$|Z^I u| \leq 2C_0 \frac{\varepsilon}{\sqrt{1+s}(1+|q|)^\delta}, \ I \leq \frac{N}{2}, \tag{3.3.2}$$

$$|Z^I u| \leq 2C_0 \frac{\varepsilon}{\sqrt{1+s}(1+|q|)^{\frac{\delta}{2}}}, \ I \leq \frac{N}{2} + 1, \tag{3.3.3}$$

$$\|w^{\frac{1}{2}} \partial Z^I u\|_{L^2} \leq 2C_0(1+t)^\rho \varepsilon, \ I \leq N. \tag{3.3.4}$$

Thanks to Klainerman-Sobolev inequality, the assumption (3.3.4) yields, for $I \leq N-2$

$$|\partial Z^I u| \lesssim \frac{\varepsilon(1+t)^\rho}{\sqrt{1+s}\sqrt{1+|q|}}, \ for\ q < 0, \quad |\partial Z^I u| \lesssim \frac{\varepsilon(1+t)^\rho}{\sqrt{1+s}(1+|q|)^{1+\delta}}, \ for\ q > 0. \tag{3.3.5}$$

and consequently, thanks to Lemma 3.1.11

$$|Z^I u| \lesssim \frac{\varepsilon\sqrt{1+|q|}}{(1+s)^{\frac{1}{2}-\rho}}, \ for\ q < 0, \quad |Z^I u| \lesssim \frac{\varepsilon}{(1+s)^{\frac{1}{2}-\rho}(1+|q|)^\delta}, \ forq > 0. \tag{3.3.6}$$

We use the $L^\infty - L^\infty$ estimate to ameliorate eqtimates (3.3.2) and (3.3.3). We write

$$\Box Z^I u = \sum_{I_1+I_2 \leq I} \partial Z^{I_1} u \bar{\partial} Z^{I_2} u. \tag{3.3.7}$$

We first treat the case $I \leq \frac{N}{2}$. We assume $I_1 \leq \frac{N}{4}$ (the case $I_2 \leq \frac{N}{4}$ can be treated in the same way). Therefore, we can estimate thanks to (3.1.14)

$$|\partial Z^{I_1} u| \leq \frac{1}{1+|q|}|Z^{I_1+1}u|.$$

Since $\frac{N}{4}+1 \leq \frac{N}{2}$ we obtain thanks to (3.3.2)

$$|\partial Z^{I_1} u| \lesssim \frac{\varepsilon}{(1+|q|)^{1+\delta}\sqrt{1+s}}.$$

To estimate $\bar{\partial} Z^{I_2} u$ we use (3.1.14) and the bootstrap assumption (3.3.3) to obtain

$$|\bar{\partial} Z^{I_2} u| \lesssim \frac{1}{1+s}|Z^{I_2+1}u| \lesssim \frac{\varepsilon}{(1+s)^{\frac{3}{2}}(1+|q|)^{\frac{\delta}{2}}}.$$

This yields

$$|\Box Z^I u| \lesssim \frac{\varepsilon^2}{(1+s)^2(1+|q|)^{1+\frac{3\delta}{2}}}.$$

We can now use the $L^\infty - L^\infty$ estimate of Proposition 3.1.10, together with the estimate of Proposition 3.1.5 and the Sobolev injection of Proposition 3.1.2, which gives

$$|Z^I u| \leq \frac{C_0\varepsilon}{\sqrt{1+s}(1+|q|)^\delta} + \frac{C\varepsilon^2\ln(1+|q|)}{\sqrt{1+s}\sqrt{1+|q|}}.$$

This implies, since $\ln(1+|q|) \lesssim (1+|q|)^{\frac{1}{2}-\delta}$

$$|Z^I u| \leq \frac{C_0\varepsilon}{\sqrt{1+s}(1+|q|)^\delta} + \frac{C\varepsilon^2}{\sqrt{1+s}(1+|q|)^\delta}. \tag{3.3.8}$$

We now treat the case $I = \frac{N}{2}+1$. We assume $I_1 \leq \frac{N+2}{4} \leq \frac{N}{2}$ so we have the same estimate as before for $\partial Z^{I_1} u$. To estimate $\bar{\partial} Z^{I_2} u$, since $\frac{N}{2}+2 \leq N-2$ we use (3.3.6). We obtain

$$|\bar{\partial} Z^{I_2} u| \lesssim \frac{1}{1+s}|Z^{I_2+1}u| \lesssim \frac{\varepsilon\sqrt{1+|q|}}{(1+s)^{\frac{3}{2}-\rho}}.$$

Therefore we obtain

$$|\Box Z^I u| \lesssim \frac{\varepsilon^2}{(1+s)^{2-\rho}(1+|q|)^{\frac{1}{2}+\delta}} \lesssim \frac{\varepsilon^2}{(1+s)^{\frac{3}{2}+\frac{\delta}{2}}(1+|q|)^{1+\frac{\delta}{2}-\rho}}.$$

Therefore, like for (3.3.8), the $L^\infty - L^\infty$ estimate yields

$$|Z^I u| \leq \frac{C_0\varepsilon}{\sqrt{1+s}(1+|q|)^\delta} + \frac{C\varepsilon^2}{\sqrt{1+s}(1+|q|)^{\frac{\delta}{2}}}. \tag{3.3.9}$$

We now use the weighted energy estimate to ameliorate (3.3.4). Let $I \leq N$. In view of (3.3.7), it implies

$$\frac{d}{dt}\|w(q)^{\frac{1}{2}}\partial Z^I u\|_{L^2}^2 + \|w'(q)^{\frac{1}{2}}\bar{\partial} Z^I u\|_{L^2}^2 \lesssim \sum_{I_1+I_2\leq I}\|w^{\frac{1}{2}}\partial Z^{I_1}u\bar{\partial}Z^{I_2}u\|_{L^2}\|w^{\frac{1}{2}}\partial Z^I u\|_{L^2}. \tag{3.3.10}$$

We first assume $I_2 \leq \frac{N}{2}$. Then we estimate

$$|\bar{\partial} Z^{I_2} u| \lesssim \frac{\varepsilon}{(1+s)^{\frac{3}{2}}(1+|q|)^{\frac{\delta}{2}}}.$$

This yields

$$\|w^{\frac{1}{2}}\partial Z^{I_1}u\bar{\partial}Z^{I_2}u\|_{L^2} \lesssim \frac{\varepsilon}{(1+t)^{\frac{3}{2}}}\|w^{\frac{1}{2}}\partial Z^{I_1}u\|_{L^2}.$$

We now assume $I_1 \leq \frac{N}{2}$. Then, we estimate

$$|\partial Z^{I_1}u| \lesssim \frac{\varepsilon}{\sqrt{1+s}(1+|q|)^{1+\frac{\delta}{2}}}.$$

Therefore we obtain

$$\|w^{\frac{1}{2}}\partial Z^{I_1}u\bar{\partial}Z^{I_2}u\|_{L^2} \lesssim \frac{\varepsilon}{\sqrt{1+t}}\left\|\frac{w^{\frac{1}{2}}}{(1+|q|)^{1+\frac{\delta}{2}}}\bar{\partial}Z^{I_2}u\right\|_{L^2}.$$

Since

$$\frac{w^{\frac{1}{2}}}{(1+|q|)^{1+\frac{\delta}{2}}} \leq w'(q)^{\frac{1}{2}},$$

we infer

$$\|w^{\frac{1}{2}}\partial Z^{I_1}u\bar{\partial}Z^{I_2}u\|_{L^2}\|w^{\frac{1}{2}}\bar{\partial}Z^{I}u\|_{L^2} \leq \frac{\varepsilon}{1+t}\|w^{\frac{1}{2}}\partial Z^{I}u\|_{L^2}^2 + \varepsilon\|w'(q)^{\frac{1}{2}}\bar{\partial}Z^{I_2}u\|_{L^2}^2.$$

Therefore (3.3.10) writes

$$\frac{d}{dt}\|w(q)^{\frac{1}{2}}\partial Z^{I}u\|_{L^2}^2 + \|w'(q)^{\frac{1}{2}}\partial\bar{Z}^{I}u\|_{L^2}^2 \lesssim \frac{\varepsilon}{1+t}\|w^{\frac{1}{2}}\partial Z^{I}u\|_{L^2}^2 + \varepsilon\|w'(q)^{\frac{1}{2}}\bar{\partial}Z^{I_2}u\|_{L^2}^2,$$

so for ε small enough

$$\frac{d}{dt}\|w(q)^{\frac{1}{2}}\partial Z^{I}u\|_{L^2}^2 + \frac{1}{2}\|w'(q)^{\frac{1}{2}}\bar{\partial}Z^{I}u\|_{L^2}^2 \lesssim \frac{\varepsilon}{1+t}\|w^{\frac{1}{2}}\partial Z^{I}u\|_{L^2}^2.$$

We obtain

$$\|w(q)^{\frac{1}{2}}\partial Z^{I}u\|_{L^2} \leq C_0\varepsilon(1+t)^{C\varepsilon}. \tag{3.3.11}$$

For ε small enough so that

$$C\varepsilon \leq \frac{C_0}{2}, \quad (1+t)^{C\varepsilon} \leq \frac{3}{2}(1+t)^{\rho},$$

we have proved, in view of (3.3.8), (3.3.9) and (3.3.11) that for $t \leq T$ we have

$$\begin{aligned}
|Z^{I}u| &\leq \frac{3}{2}C_0\frac{\varepsilon}{\sqrt{1+s}(1+|q|)^{\delta}},\ |I| \leq \frac{N}{2},\\
|Z^{I}u| &\leq \frac{3}{2}C_0\frac{\varepsilon}{\sqrt{1+s}(1+|q|)^{\frac{\delta}{2}}},\ |I| \leq \frac{N}{2}+1,\\
\|w^{\frac{1}{2}}\partial Z^{I}u\|_{L^2} &\leq \frac{3}{2}C_0(1+t)^{\rho}\varepsilon,\ |I| \leq N,
\end{aligned}$$

which concludes the proof. □

Remark 3.3.2. *Actually, only the highest order energy* $\|w^{\frac{1}{2}}\partial Z^{N}u\|_{L^2}$ *grows in* t*. To see this, we estimate*

$$\|w^{\frac{1}{2}}\partial Z^{I_1}u\bar{\partial}Z^{I_2}u\|_{L^2}$$

for $I_1 \leq \frac{N}{2}$ *and* $I_2 \leq N-1$*. Since*

$$|\bar{\partial}Z^{I_2}u| \leq \frac{1}{1+s}|Z^{I_2+1}|,$$

we obtain, together with the weighted Hardy inequality

$$\|w^{\frac{1}{2}}\partial Z^{I_1}u\bar{\partial}Z^{I_2}u\|_{L^2}\lesssim\frac{\varepsilon}{(1+t)^{\frac{3}{2}}}\left\|\frac{w^{\frac{1}{2}}}{(1+|q|)}Z^{I_2+1}u\right\|_{L^2}\lesssim\frac{\varepsilon}{(1+t)^{\frac{3}{2}}}\|w^{\frac{1}{2}}\partial Z^{I_2+1}u\|_{L^2}.$$

Therefore, the weighted energy estimate yields, for $|I|\leq N-1$

$$\frac{d}{dt}\|w^{\frac{1}{2}}\partial Z^I u\|^2_{L^2}\lesssim\frac{\varepsilon^2}{(1+t)^{\frac{3}{2}-C\varepsilon}},$$

and hence

$$\|w^{\frac{1}{2}}\partial Z^I u\|_{L^2}\lesssim\varepsilon.$$

Remark 3.3.3. *The use of the term $\|w'(q)^{\frac{1}{2}}\bar{\partial}Z^I u\|^2_{L^2}$ to exploit the structure in the energy estimate has been introduced by Alinhac in [1] and is sometimes called Alinhac ghost weight method. It has also been used in the case of Einstein equations in wave coordinates in [40].*

Unfortunately, Einstein equations do not have the null structure, but only a weak form of it. In the next sections, we will see what problems this creates and the method we used to tackle them. We will be less precise than in this first example, since full details will be provided when we proceed with the proof of Theorem 3.1.12.

3.3.2 The coefficient $g_{\underline{L}\underline{L}}$

To understand how to deal with $g_{\underline{L}\underline{L}}$, let us consider the question of global existence for the following system, which is of the form (3.2.6)

$$\begin{cases}\Box\varphi=0,\\ \Box h=-2(\partial_q\varphi)^2-2\frac{b(\theta)\partial^2_q(q\chi(q))}{r}.\end{cases}\tag{3.3.12}$$

with initial data for φ of size ε and zero initial data for h. We recall $\|b\|_{L^2(\mathbb{S}^1)}\lesssim\varepsilon^2$. We have the following estimates for φ

$$\|w^{\frac{1}{2}}\partial\varphi\|_{L^2}\lesssim\varepsilon,\quad|\partial\varphi|\lesssim\frac{\varepsilon}{\sqrt{1+s}(1+|q|)^{1+\delta}}.$$

Therefore, the energy estimate for h writes

$$\frac{d}{dt}\|w^{\frac{1}{2}}\partial h\|^2_{L^2}\lesssim\left(\|w^{\frac{1}{2}}(\partial_q\varphi)^2\|_{L^2}+\left\|w^{\frac{1}{2}}\frac{b(\theta)\partial^2_q(q\chi(q))}{r}\right\|_{L^2}\right)\|w^{\frac{1}{2}}\partial h\|_{L^2},$$

and thus

$$\frac{d}{dt}\|w^{\frac{1}{2}}\partial h\|_{L^2}\lesssim\left(\frac{\varepsilon}{\sqrt{1+t}}\|w^{\frac{1}{2}}\partial\varphi\|_{L^2}+\frac{\varepsilon^2}{\sqrt{1+t}}\right)\lesssim\frac{\varepsilon^2}{\sqrt{1+t}}.$$

We infer

$$\|w^{\frac{1}{2}}\partial h\|_{L^2}\leq\varepsilon^2\sqrt{1+t}.\tag{3.3.13}$$

This estimate is not sufficient. To obtain more information on h, we will approximate it by the solution h_0 of the following transport equation (this procedure will be made more precise in Section 3.7)

$$\partial_q h_0=-2r(\partial_q\varphi)^2-2b(\theta)\partial^2_q(q\chi(q)),\tag{3.3.14}$$

with initial data $h_0=0$ at $t=0$. The L^∞ estimate for φ, and the fact that χ' is supported in $[1,2]$ yield

$$|\partial_q h_0|\lesssim\frac{\varepsilon^2}{(1+|q|)^{2+2\delta}}.$$

To estimate h_0 we write

$$h_0(Q,s,\theta) = \int_s^Q \left(-2(\partial_q\varphi)^2 r - 2b(\theta)\partial_q^2(q\chi(q))\right) dq,$$

so we obtain

$$h_0(s,Q,\theta) = O\left(\frac{\varepsilon^2}{(1+|Q|)^{1+2\delta}}\right),\ Q>0,$$
$$h_0(s,Q,\theta) = \int_s^{-s} \left(-2(\partial_q\varphi)^2 r - 2b(\theta)\partial_q^2(q\chi(q))\right) dq + O\left(\frac{\varepsilon^2}{(1+|Q|)^{1+2\delta}}\right),\ Q<0.$$

Therefore, since

$$\int_s^{-s} \partial_q^2(q\chi(q))dq = -1,\ for\ s\geq 2$$

to maximize the decay in q for h_0 (and hence for h, provided one has a suitable control over $h-h_0$) we will choose b such that

$$b(\theta) \simeq \int_s^{-s} (\partial_q\varphi)^2 r dq. \tag{3.3.15}$$

Remark 3.3.4. *$b(\theta)$ is a free parameter, except from $\int b(\theta)$, $\int b(\theta)\cos(\theta)$ and $\int b(\theta)\sin(\theta)$ which are prescribed by the resolution of the constraint equations, and correspond intuitively to the ADM angle (energy) and linear momentum. Let Π be the projection defined by* (3.12.7)*. Then*

$$\Pi(b(\theta)) \simeq \Pi\left(\int_s^{-s} (\partial_q\varphi)^2 r dq\right),$$

will be forced in the course of the bootstrap procedure. On the other hand, the fact that

$$\int b(\theta) \simeq \int\int_s^{-s} (\partial_q\varphi)^2 r dq d\theta,$$
$$\int b(\theta)\cos(\theta) \simeq \int\int_s^{-s} (\partial_q\varphi)^2 \cos(\theta) r dq d\theta,$$
$$\int b(\theta)\sin(\theta) \simeq \int\int_s^{-s} (\partial_q\varphi)^2 \sin(\theta) r dq d\theta.$$

will be obtained by integrating the constraint equations at any time t (see Section 3.7).

3.3.3 Non commutation of the wave operator with the null frame

In this section, we will discuss the influence of the terms appearing in (3.2.7). We have seen in the previous section that h_0 does not decay at all with respect to the s variable. In turn, we will show that this is also the case for h, and finally for the coefficient $g_{\underline{LL}}$. We do not want this behavior to propagate to the other coefficients of the metric. To this end, we will rely on a decomposition of the type

$$g = g_b + \Upsilon\left(\frac{r}{t}\right)\frac{g_{\underline{LL}}}{4}dq^2 + \widetilde{g}_i.$$

However, since the wave operator does not commute with the null decomposition, we have to control the solution $\widetilde{g}_i$ of an equation of the form

$$\Box\widetilde{g}_i = \Upsilon\left(\frac{r}{t}\right)\frac{\bar\partial h}{r},$$

where h is the solution of (3.3.12). The term $\Upsilon\left(\frac{r}{t}\right)\frac{\bar{\partial}h}{r}$ has the form of the terms appearing in (3.2.7).

Provided we can approximate h by the solution h_0 of the transport equation (3.3.14), we obtain decay with respect to q for h. The decay we will be able to get is

$$|h| \lesssim \frac{\varepsilon^2}{\sqrt{1+|q|}}.$$

With this decay we infer

$$|\Box \widetilde{g}_i| \lesssim \frac{\varepsilon^2}{(1+s)^2\sqrt{1+|q|}},$$

and therefore, with the $L^\infty - L^\infty$ estimate, we deduce

$$|\widetilde{g}_i| \lesssim \frac{\varepsilon}{(1+s)^{\frac{1}{2}-\rho}},$$

for all $\rho > 0$.

On the other hand, assume we are only allowed to use the energy estimate for h, which is the case when deriving L^2 type estimates for $\widetilde{g}_i$ at the level of the highest energy. When applying the weighted energy estimate for $\widetilde{g}_i$, we obtain

$$\frac{d}{dt}\|w(q)^{\frac{1}{2}}\partial\widetilde{g}_i\|^2_{L^2} \leq \left\|w(q)^{\frac{1}{2}}\Upsilon\left(\frac{r}{t}\right)\frac{\bar{\partial}h}{r}\right\|_{L^2}\|w(q)^{\frac{1}{2}}\partial\widetilde{g}_i\|.$$

We estimate

$$\left\|w(q)^{\frac{1}{2}}\Upsilon\left(\frac{r}{t}\right)\frac{\bar{\partial}h}{r}\right\|_{L^2} \lesssim \frac{1}{1+t}\|w(q)^{\frac{1}{2}}\partial h\|_{L^2} \lesssim \frac{\varepsilon^2}{\sqrt{1+t}}, \tag{3.3.16}$$

where we have used the estimate (3.3.13) of the previous section for h. This yields

$$\frac{d}{dt}\|w(q)^{\frac{1}{2}}\partial\widetilde{g}_i\|_{L^2} \leq \frac{\varepsilon^2}{\sqrt{1+t}}.$$

So

$$\|w(q)^{\frac{1}{2}}\partial\widetilde{g}_i\|_{L^2} \leq \varepsilon^2\sqrt{1+t},$$

which is precisely the behaviour we are trying to avoid with this decomposition ! However we have not been able to exploit all the structure in (3.3.16). In order to do so, we will use different weight functions for $\widetilde{g}_i$ and for h. If we set

$$\widetilde{w}(q) = (1+|q|)^{1+2\mu}w(q),$$

with $0 < \mu \leq \frac{1}{4}$ and we assume that we have

$$\|\widetilde{w}(q)^{\frac{1}{2}}\partial h\|_{L^2} \lesssim \varepsilon^2\sqrt{1+t},$$

then we can estimate

$$\left\|w(q)^{\frac{1}{2}}\Upsilon\left(\frac{r}{t}\right)\frac{\bar{\partial}h}{r}\right\|_{L^2} \lesssim \frac{1}{1+t}\left\|\widetilde{w}(q)^{\frac{1}{2}}\Upsilon\left(\frac{r}{t}\right)\frac{\bar{\partial}h}{(1+|q|)^{\frac{1}{2}+\mu}}\right\|_{L^2}.$$

We write

$$|\bar{\partial}h| \lesssim \frac{1}{1+s}|Zh| \lesssim \frac{1}{(1+s)^{\frac{1}{2}+\mu}(1+|q|)^{\frac{1}{2}-\mu}}|Zh|,$$

so we obtain

$$\left\|w(q)^{\frac{1}{2}}\Upsilon\left(\frac{r}{t}\right)\frac{\bar{\partial}h}{r}\right\|_{L^2} \lesssim \frac{1}{(1+t)^{\frac{3}{2}+\mu}}\left\|\widetilde{w}(q)^{\frac{1}{2}}\frac{Zh}{1+|q|}\right\|_{L^2} \lesssim \frac{1}{(1+t)^{\frac{3}{2}+\mu}}\|\widetilde{w}(q)^{\frac{1}{2}}\partial Zh\|_{L^2},$$

where we used the weighted Hardy inequality. Consequently, the energy inequality for $\widetilde{g}_i$ yields

$$\frac{d}{dt}\|w(q)^{\frac{1}{2}}\partial\widetilde{g}_i\|_{L^2} \lesssim \frac{\varepsilon^2}{(1+t)^{1+\mu}},$$

and therefore

$$\|w(q)^{\frac{1}{2}}\partial\widetilde{g}_i\|_{L^2} \lesssim \varepsilon^2.$$

Recall that the weighted energy inequality forbids weights of the form $(1+|q|)^\alpha$ with $\alpha > 0$ in the region $q < 0$. Therefore we are forced to make the following choice in the region $q < 0$

$$\widetilde{w}(q) = O(1), \quad w(q) = \frac{1}{(1+|q|)^{1+2\mu}}.$$

Thus, for $\widetilde{g}_i$, the $\sqrt{t}$ loss has been replaced by a loss in $(1+|q|)^{\frac{1}{2}+\mu}$.

3.3.4 The quasilinear structure

In this section we will discuss the challenges of the quasilinear structure. We will take as an example the equation for φ, $\Box_g\varphi = 0$. Following Remark 3.2.2, we can focus on the terms of the form $g_{LL}\partial_q^2\varphi$. The wave coordinates condition yields

$$\partial g_{LL} \sim \bar{\partial} g.$$

If g satisfied $\Box g = 0$, the L^∞ estimates for g given by Corollary 3.1.6 for suitable initial data would imply

$$|\partial g_{LL}| \leq \frac{\varepsilon}{(1+s)^{\frac{3}{2}}\sqrt{1+|q|}},$$

We would like to keep this decay in $\frac{1}{(1+s)^{\frac{3}{2}}}$ after integrating with respect to q. However, we are not in the range of application of Lemma 3.1.11. To this end, we will assume more decay on the initial data. As stated in Theorem 3.1.12, we take $(g, \partial_t g) \in H^{N+1}_\delta \times H^N_{\delta+1}$ with $\frac{1}{2} < \delta < 1$. Then, with the weight w_0 stated in Theorem 3.1.12, the weighted energy inequality yields

$$\|w_0(q)\partial Z g\|_{L^2} \lesssim \varepsilon,$$

and consequently, for $q > 0$, the weighted Klainerman-Sobolev inequality yields

$$|\partial Z g| \lesssim \frac{\varepsilon}{\sqrt{1+s}(1+|q|)^{\frac{3}{2}+\delta}}.$$

If we integrate from $t = 0$, we obtain for $q > 0$

$$|Zg| \lesssim \frac{\varepsilon}{\sqrt{1+s}(1+|q|)^{\frac{1}{2}+\delta}}.$$

By writing $|\bar{\partial} g| \lesssim \frac{1}{1+s}|Zg|$, we obtain

$$|\partial g_{LL}| \lesssim \frac{\varepsilon}{(1+s)^{\frac{3}{2}}(1+|q|)^{\frac{1}{2}}}, \; for\; q<0, \quad |\partial g_{LL}| \lesssim \frac{\varepsilon}{(1+s)^{\frac{3}{2}}(1+|q|)^{\frac{1}{2}+\delta}}, \; for\; q>0.$$

Since $\delta > \frac{1}{2}$ we can apply Lemma 3.1.11, which yields

$$|g_{LL}| \lesssim \frac{\varepsilon\sqrt{1+|q|}}{(1+s)^{\frac{3}{2}}}, \; for q<0, \quad |g_{LL}| \lesssim \frac{\varepsilon}{(1+s)^{\frac{3}{2}}(1+|q|)^{\delta-\frac{1}{2}}}, \; for\; q>0.$$

Consequently we easily estimate

$$\|w^{\frac{1}{2}} g_{LL}\partial_q^2 Z^I\varphi\|_{L^2} \lesssim \frac{\varepsilon}{(1+t)^{\frac{3}{2}}}\|w^{\frac{1}{2}}\partial_q Z^{I+1}\varphi\|_{L^2}.$$

This strong decay in the region $q>0$ is also needed when estimating

$$\|w_0^{\frac{1}{2}} Z^I g_{LL}\partial_q^2\varphi\|_{L^2}.$$

The idea will be first to use the weighted Hardy inequality to derive

$$\|w_0^{\frac{1}{2}} Z^I g_{LL}\partial_q^2\varphi\|_{L^2} \lesssim \frac{\varepsilon}{\sqrt{1+t}}\left\|\frac{w_0^{\frac{1}{2}}}{(1+|q|)^2} Z^I g_{LL}\right\|_{L^2} \lesssim \frac{\varepsilon}{\sqrt{1+t}}\left\|\frac{w_0^{\frac{1}{2}}}{(1+|q|)}\partial Z^I g_{LL}\right\|_{L^2}.$$

Then we rely on the wave coordinates condition, which yields

$$|\partial Z^I g_{LL}| \lesssim |\bar{\partial} Z^I g| \lesssim \frac{1}{1+s}|Z^{I+1}g|,$$

and then use the weighted Hardy inequality again. However, one has to be careful when using the weighted Hardy inequality. In the region $q>0$ the weight must be sufficiently large to allow to perform it twice. This is an other reason why we work with initial data in H^N_δ with $\delta>\frac{1}{2}$, which is more than the decay which is necessary to prove the global well posedness of a semi linear wave equation with null structure.

3.3.5 Interaction with the metric g_b

In this section we want to discuss the influence of the crossed terms between g_b and $\varphi,\widetilde{g}$. We will take as an example the equation for φ, $\Box_g\varphi=0$. As discussed in Section 3.2.5, we can focus on the term (3.2.13). We may look at the following model problem

$$\Box\varphi = \frac{\varepsilon}{r}\chi(q)\bar{\partial}\varphi.$$

If we perform the weighted energy estimate, we obtain

$$\frac{d}{dt}\|w_0(q)^{\frac{1}{2}}\partial Z^I\varphi\|^2 + \|w_0'(q)^{\frac{1}{2}}\bar{\partial} Z^I\varphi\|^2_{L^2} \lesssim \frac{\varepsilon}{1+t}\|w_0^{\frac{1}{2}}\partial Z^I\varphi\|^2_{L^2}.$$

Therefore

$$\|w_0(q)^{\frac{1}{2}}\partial Z^I\varphi\|_{L^2} \leq C_0\varepsilon(1+t)^{C\varepsilon}$$

and for all $\sigma>0$

$$\int_0^T \frac{1}{(1+t)^\sigma}\|w_0'(q)^{\frac{1}{2}}\bar{\partial} Z^I\varphi\|^2_{L^2}dt \lesssim \varepsilon^2. \tag{3.3.17}$$

To avoid this logarithmic loss, we need to exploit more the structure of the equation. To this end we introduce the weight modulator

$$\begin{cases} \alpha(q) = \frac{1}{(1+|q|)^\sigma},\ q>0,\\ \alpha(q)=1,\ q<0,\end{cases}$$

for $0<\sigma<\frac{1}{2}$. Then the energy inequality yields

$$\frac{d}{dt}\|\alpha w_0(q)^{\frac{1}{2}}\partial Z^I\varphi\|^2_{L^2} \leq \varepsilon\left\|\mathbb{1}_{q>0}\frac{\alpha w_0^{\frac{1}{2}}}{1+s}\bar{\partial} Z^I\varphi\right\|_{L^2}\|\alpha w_0(q)^{\frac{1}{2}}\partial Z^I\varphi\|_{L^2}.$$

We estimate, for $q>0$

$$\frac{\alpha(q)}{1+s} \lesssim \frac{1}{(1+t)^{\frac{1}{2}+\sigma}(1+|q|)^{\frac{1}{2}}}.$$

And therefore, we obtain

$$\begin{aligned}\frac{d}{dt}\|\alpha w_0(q)^{\frac12}\partial Z^I\varphi\|_{L^2}^2 &\lesssim \frac{\varepsilon}{(1+t)^{\frac12+\sigma}}\left\|\mathbb{1}_{q>0}\frac{w_0^{\frac12}}{\sqrt{1+|q|}}\bar\partial Z^I\varphi\right\|_{L^2}\|\alpha w_0(q)^{\frac12}\partial Z^I\varphi\|_{L^2}\\ &\lesssim \frac{\varepsilon}{(1+t)^{\sigma}}\|w_0'(q)^{\frac12}\bar\partial Z^I\varphi\|_{L^2}+\frac{\varepsilon}{(1+t)^{1+\sigma}}\|\alpha w_0(q)^{\frac12}\partial Z^I\varphi\|_{L^2}.\end{aligned}$$

and consequently in view of (3.3.17) we obtain

$$\|\alpha w_0(q)^{\frac12}\partial Z^I\varphi\|_{L^2}\le C_0\varepsilon+C\varepsilon^2.$$

With this technique, the logarithmic loss in t has been replaced by a small loss in q.

3.4 Bootstrap assumptions and proof of Theorem 3.1.12

3.4.1 Bootstrap assumptions

Let $\frac12<\delta<1$. In view of the assumptions of Theorem 3.1.12, the initial data (φ_0,φ_1) for φ are given in $H^{N+1}_\delta(\mathbb{R}^2)\times H^N_{\delta+1}(\mathbb{R}^2)$.

For $\widetilde b\in W^{2,N}$ such that

$$\int_{\mathbb{S}^1}\widetilde b=\int_{\mathbb{S}^1}\widetilde b\cos(\theta)=\int_{\mathbb{S}^1}\widetilde b\sin(\theta)=0,$$

and

$$\|\widetilde b\|_{W^{2,N}}\lesssim 2C_0\varepsilon^2,$$

Theorem 3.1.3 allows us to find initial data g and $\partial_t g$ such that

- g_{ij}, K_{ij} satisfy the constraint equations,
- g and $\partial_t g$ are compatible with the decomposition $g=g_b+\widetilde g$, where

$$b(\theta)=\widetilde b(\theta)+b_0+b_1\cos(\theta)+b_2\sin(\theta) \tag{3.4.1}$$

  with $b_0,b_2,b_2,J(\theta)$ given by Theorem 3.1.3,
- the generalized wave coordinate condition given by H_b is satisfied at $t=0$.

The system 3.2.1 being a standard quasilinear system of wave equations, we know that there exists a solution until a time T. Moreover with our conditions on the initial data, our solution (g,φ) is solution of the Einstein equations (3.1.1), and the wave coordinate condition is satisfied for $t\le T$ (see Appendix 3.12.2).

Remark 3.4.1. *Our choice of generalized wave coordinates does not change the hyperbolic structure because H_b does not contain derivatives of $\widetilde g$.*

We take three parameters ρ,σ,μ such that

$$0<\rho\ll\sigma\ll\mu\ll\delta, \tag{3.4.2}$$

$$\sigma+\rho<\delta-\frac12. \tag{3.4.3}$$

We consider a time $T>0$ such that there exists $b(\theta)\in W^{N,2}(\mathbb{S}^1)$ and a solution $(\varphi,\widetilde g)$ of (3.2.4) on $[0,T]$, associated to initial data for g. We assume that on $[0,T]$, the following estimates hold.

Bootstrap assumptions for b

$$\left\| \partial_\theta^I \left(\Pi b(\theta) + \Pi \int_{\Sigma_{T,\theta}} (\partial_q \varphi)^2 r dq \right) \right\|_{L^2(\mathbb{S}^1)} \leq B \frac{\varepsilon^2}{\sqrt{T}}, \; for \; I \leq N-4 \tag{3.4.4}$$

$$\|\partial_\theta^I b(\theta)\|_{L^2(\mathbb{S}^1)} \leq B\varepsilon^2, \; for \; I \leq N \tag{3.4.5}$$

where Π is the projection defined by (3.12.7), $\int_{\Sigma_{T,\theta}}$ is defined by (3.1.15) and B is a constant depending on $\rho, \sigma, \mu, \delta, N$.

We introduce four decomposition of the metric g

$$g = g_b + \Upsilon\left(\frac{r}{t}\right) h_0 dq^2 + \widetilde{g}_1, \tag{3.4.6}$$

$$g = g_b + \Upsilon\left(\frac{r}{t}\right) (h_0 + \widetilde{h}) dq^2 + \widetilde{g}_2, \tag{3.4.7}$$

$$g = g_b + \Upsilon\left(\frac{r}{t}\right) h dq^2 + \widetilde{g}_3, \tag{3.4.8}$$

$$g = g_b + \Upsilon\left(\frac{r}{t}\right) h dq^2 + \Upsilon\left(\frac{r}{t}\right) k r dq d\theta + \widetilde{g}_4, \tag{3.4.9}$$

where h_0 is the solution of the transport equation

$$\begin{cases} \partial_q h_0 = -2r(\partial_q \varphi)^2 - 2b(\theta)\partial_q^2(\chi(q)q), \\ h_0|_{t=0} = 0, \end{cases} \tag{3.4.10}$$

$\widetilde{h}$ is solution of the linear wave equation

$$\begin{cases} \Box \widetilde{h} = \Box\left(\Upsilon\left(\frac{r}{t}\right) h_0\right) + \Upsilon\left(\frac{r}{t}\right) g_{LL} \partial_q^2 h_0 + 2\Upsilon\left(\frac{r}{t}\right)(\partial_q\varphi)^2 - 2(R_b)_{qq} + \Upsilon\left(\frac{r}{t}\right) \widetilde{Q}_{\underline{L}\underline{L}}(h_0, \widetilde{g}), \\ (\widetilde{h}, \partial_t \widetilde{h})|_{t=0} = (0,0), \end{cases} \tag{3.4.11}$$

where

$$\widetilde{Q}_{\underline{L}\underline{L}}(h_0, \widetilde{g}) = \partial_{\underline{L}} g_{LL} \partial_{\underline{L}} h_0 + \partial_{\underline{L}} (g_b)_{UU} \partial_{\underline{L}} g_{\underline{L}\underline{L}}. \tag{3.4.12}$$

$$\begin{cases} \Box_g h = -2(\partial_q\varphi)^2 + 2(R_b)_{qq} + Q_{\underline{L}\underline{L}}(h, \widetilde{g}), \\ (k, \partial_t k)|_{t=0} = (0,0), \end{cases} \tag{3.4.13}$$

and k is the solution of

$$\begin{cases} \Box_g k = \partial_U g_{LL} \partial_q h, \\ (h, \partial_t h)|_{t=0} = (0,0). \end{cases} \tag{3.4.14}$$

L^∞-based bootstrap assumptions For $I \leq N-14$ we assume

$$|Z^I \varphi| \leq \frac{2C_0\varepsilon}{\sqrt{1+s}(1+|q|)^{\frac{1}{2}-4\rho}}, \tag{3.4.15}$$

$$|Z^I \widetilde{g}_1| \leq \frac{2C_0\varepsilon}{(1+s)^{\frac{1}{2}-\rho}}, \tag{3.4.16}$$

where here and in the following, C_0 is a constant depending on $\rho, \sigma, \mu, \delta, N$ such that the inequalities are satisfied at $t=0$ with $2C_0$ replaced by C_0. For $I \leq N-12$ we assume

$$|Z^I \varphi| \leq \frac{2C_0\varepsilon}{(1+s)^{\frac{1}{2}-2\rho}}, \tag{3.4.17}$$

$$|Z^I \widetilde{g}_1| \leq \frac{2C_0\varepsilon}{(1+s)^{\frac{1}{2}-2\rho}}. \tag{3.4.18}$$

We assume the following estimate for h_0 for $I \leq N-7$ and $q<0$

$$|Z^I h_0| \leq \frac{2C_0\varepsilon}{(1+s)^{\frac{1}{2}}} + \frac{2C_0\varepsilon}{(1+|q|)^{1-4\rho}}. \tag{3.4.19}$$

and for $q>0$

$$|Z^I h_0| \leq \frac{2C_0\varepsilon}{(1+|q|)^{2+2(\delta-\sigma)}}. \tag{3.4.20}$$

We also assume the following for $\widetilde{h}$ and $I \leq N-7$

$$|Z^I \widetilde{h}| \leq \frac{2C_0\varepsilon}{(1+|q|)^{\frac{1}{2}-\rho}}. \tag{3.4.21}$$

L^2-based bootstrap assumptions We introduce four weight functions

$$\begin{cases} w_0(q) = (1+|q|)^{2+2\delta}, \ q>0, \\ w_0(q) = 1 + \frac{1}{(1+|q|)^{2\mu}}, \ q<0, \end{cases}$$

$$\begin{cases} w_1(q) = (1+|q|)^{2+2\delta}, \ q>0, \\ w_1(q) = \frac{1}{(1+|q|)^{\frac{1}{2}}}, \ q<0, \end{cases}$$

$$\begin{cases} w_2(q) = (1+|q|)^{2+2\delta}, \ q>0, \\ w_2(q) = \frac{1}{(1+|q|)^{1+2\mu}}, \ q<0, \end{cases}$$

$$\begin{cases} w_3(q) = (1+|q|)^{3+2\delta}, \ q>0, \\ w_3(q) = 1 + \frac{1}{(1+|q|)^{2\mu}}, \ q<0. \end{cases}$$

We also introduce weight modulators

$$\begin{cases} \alpha(q) = \frac{1}{(1+|q|)^{\sigma}}, \ q>0, \\ \alpha(q) = 1, \ q<0, \end{cases} \tag{3.4.22}$$

$$\begin{cases} \alpha_2(q) = \frac{1}{(1+|q|)^{2\sigma}}, \ q>0, \\ \alpha_2(q) = 1, \ q<0. \end{cases}$$

We assume the following estimate for $I \leq N$

$$\begin{aligned} &\|\alpha_2 w_0(q)^{\frac{1}{2}} \partial Z^I \varphi\|_{L^2} + \|\alpha_2 w_2(q)^{\frac{1}{2}} \partial Z^I \widetilde{g}_4\|_{L^2} \\ &+ \frac{1}{\sqrt{1+t}} \|\alpha_2(q)^{\frac{1}{2}} \partial Z^I h\|_{L^2} + \frac{1}{\sqrt{1+t}} \|\alpha_2 w_3(q)^{\frac{1}{2}} \partial Z^I k\|_{L^2} \leq 2C_0\varepsilon(1+t)^{\rho}. \end{aligned} \tag{3.4.23}$$

for $I \leq N-1$

$$\|w_0(q)^{\frac{1}{2}} \partial Z^I \varphi\|_{L^2} + \|w_2(q)^{\frac{1}{2}} \partial Z^I \widetilde{g}_3\|_{L^2} + \frac{1}{\sqrt{1+t}} \|w_3(q)^{\frac{1}{2}} \partial Z^I h\|_{L^2} \leq 2C_0\varepsilon(1+t)^{\rho} \tag{3.4.24}$$

and for $I \leq N-2$

$$\|\alpha(q) w_0(q)^{\frac{1}{2}} \partial Z^I \varphi\|_{L^2} + \|\alpha(q) w_2(q)^{\frac{1}{2}} \partial Z^I \widetilde{g}_3\|_{L^2} + \frac{1}{\sqrt{1+t}} \|\alpha(q) w_3(q)^{\frac{1}{2}} \partial Z^I h\|_{L^2} \leq 2C_0\varepsilon. \tag{3.4.25}$$

In addition, for $I \leq N-8$ we assume

$$\|w_1(q)^{\frac{1}{2}} \partial Z^I \widetilde{g}_2\|_{L^2} \leq 2C_0\varepsilon(1+t)^{\rho}, \quad \|\alpha(q) w_1(q)^{\frac{1}{2}} \partial Z^I \widetilde{g}_2\|_{L^2} \leq 2C_0\varepsilon \tag{3.4.26}$$

and for $I \leq N-9$ we assume

$$\|w_0(q)^{\frac{1}{2}} \partial Z^I \widetilde{g}_2\|_{L^2} \leq 2C_0\varepsilon(1+t)^{\rho}, \quad \|\alpha(q) w_0(q)^{\frac{1}{2}} \partial Z^I \widetilde{g}_2\|_{L^2} \leq 2C_0\varepsilon. \tag{3.4.27}$$

Let us do two remarks to justify our different decompositions of the metric, and our different weight functions.

Remark 3.4.2. *We use the decomposition* (3.4.6) *instead of* (3.4.7) *to avoid a logarithmic loss when we want to improve* (3.4.16) *and* (3.4.18) *with the* $L^\infty - L^\infty$ *estimate. This loss would have been due to the terms coming from the non commutation of the wave operator with the null decomposition* (3.4.7)*. However, we use the decomposition* (3.4.7) *instead of* (3.4.6) *to avoid a logarithmic loss in the energy estimate due to the term* $\widetilde{Q}_{\underline{L}\underline{L}}$.

When h_0 *is a good approximation for* h*, we use the decomposition* (3.4.7) *instead of* (3.4.8) *in the energy estimate. This allow us to have a better control on the terms coming from the non commutation of the wave operator with the null decomposition. When* h_0 *is no longer a good approximation for* h*, we use the decomposition* (3.4.8)*. Finally, the decomposition* (3.4.9) *allow us to isolate the term* $Z^N \partial_U g_{LL} \partial_{\underline{L}} g_{\underline{L}\underline{L}}$ *on which we do not have a good control.*

Remark 3.4.3. *The weight* w_2 *is introduced to deal with the non commutation of the wave operator with the null decomposition (see Section 3.3.3). The weight* w_1 *is a transition weight between* w_0 *and* w_2*. The weight* w_3 *allows us to compensate the loss in* $\sqrt{1+t}$ *for* $g_{\underline{L}\underline{L}}$ *by an additional decay in* $\sqrt{1+|q|}$ *in the exterior region.*

The weight modulators α_1 *and* α_2 *are introduced to transform the logarithmic loss due to the interaction with the metric* g_b *in a small loss in* q *(see Section 3.3.5).*

3.4.2 Proof of Theorem 3.1.12

We have the following improvement for the bootstrap assumptions. The constant C will denote a constant depending only on $\rho, \sigma, \mu, \delta, N$. The proof of Proposition 3.4.4 is the object of Section 3.7.

Proposition 3.4.4. *Let* $I \le N-5$*. We have the estimates*

$$|Z^I h_0| \le \frac{C\varepsilon^2}{\sqrt{1+s}} + \frac{C\varepsilon^2}{(1+|q|)^{1-4\rho}},\ for\ q<0, \quad |Z^I h_0| \le \frac{C\varepsilon^2}{(1+|q|)^{2+2(\delta-\sigma)}},\ for\ q>0.$$

Let $I \le N-7$*. We have the estimate*

$$|Z^I \widetilde{h}| \le \frac{C\varepsilon^2}{(1+s)^{\frac{1}{2}-\rho}}.$$

The proof of Proposition 3.4.5 is the object of Section 3.8.

Proposition 3.4.5. *Let* $I \le N-14$*. We have the estimates*

$$|Z^I \widetilde{g}_1| \le \frac{C_0\varepsilon + C\varepsilon^2}{(1+s)^{\frac{1}{2}-\rho}}, \quad |Z^I \varphi| \le \frac{C_0\varepsilon + C\varepsilon^2}{\sqrt{1+s}(1+|q|)^{\frac{1}{2}-4\rho}}.$$

Let $I \le N-12$*. We have the estimates*

$$|Z^I \varphi| \le \frac{C_0\varepsilon + C\varepsilon^2}{(1+s)^{\frac{1}{2}-2\rho}}, \quad |Z^I \widetilde{g}_1| \lesssim \frac{C_0\varepsilon + C\varepsilon^2}{(1+s)^{\frac{1}{2}-2\rho}}.$$

The proof of Proposition 3.4.6 is the object of Section 3.10.

Proposition 3.4.6. *We have the estimates for* $I \le N$

$$\|\alpha_2 w_0(q)^{\frac{1}{2}} \partial Z^I \varphi\|_{L^2} + \|\alpha_2 w_2(q)^{\frac{1}{2}} \partial Z^I \widetilde{g}_4\|_{L^2} \le (C_0\varepsilon + \varepsilon)(1+t)^{C\sqrt{\varepsilon}},$$

$$\|\alpha_2(q)^{\frac{1}{2}} \partial Z^I h\|_{L^2} + \|\alpha_2 w_3(q)^{\frac{1}{2}} \partial Z^I k\|_{L^2} \le C\varepsilon^2 (1+t)^{\frac{1}{2}+C\sqrt{\varepsilon}},$$

for $I \le N-1$

$$\|w_0(q)^{\frac{1}{2}} \partial Z^I \varphi\|_{L^2} + \|w_2(q)^{\frac{1}{2}} \partial Z^I \widetilde{g}_3\|_{L^2} \le (C_0\varepsilon + \varepsilon)(1+t)^{C\sqrt{\varepsilon}},$$

$$\|w_3(q)^{\frac{1}{2}}\partial Z^I h\|_{L^2} \leq C\varepsilon^2(1+t)^{\frac{1}{2}+C\sqrt{\varepsilon}},$$

for $I \leq N-2$

$$\|\alpha(q)w_0(q)^{\frac{1}{2}}\partial Z^I \varphi\|_{L^2} + \|\alpha(q)w_2(q)^{\frac{1}{2}}\partial Z^I \widetilde{g}_3\|_{L^2} \leq C_0\varepsilon + C\varepsilon^{\frac{5}{4}},$$

$$\|\alpha(q)w_3(q)^{\frac{1}{2}}\partial Z^I h\|_{L^2} \leq C\varepsilon^{\frac{3}{2}},$$

for $I \leq N-7$

$$\|w_1(q)^{\frac{1}{2}}\partial Z^I \widetilde{g}_2\|_{L^2} \leq C_0\varepsilon(1+t)^{C\sqrt{\varepsilon}} + \varepsilon, \quad \|\alpha(q)w_1(q)^{\frac{1}{2}}\partial Z^I \widetilde{g}_2\|_{L^2} \leq C_0\varepsilon + C\varepsilon^{\frac{5}{4}},$$

and for $I \leq N-8$

$$\|w_0(q)^{\frac{1}{2}}\partial Z^I \widetilde{g}_2\|_{L^2} \leq C_0\varepsilon(1+t)^{C\sqrt{\varepsilon}} + \varepsilon, \quad \|\alpha(q)w_0(q)^{\frac{1}{2}}\partial Z^I \widetilde{g}_2\|_{L^2} \leq C_0\varepsilon + C\varepsilon^{\frac{5}{4}}.$$

The proof of Proposition 3.4.7 is the object of Section 3.11

Proposition 3.4.7. *We assume that the time T satisfies*

$$T \leq \exp\left(\frac{C}{\sqrt{\varepsilon}}\right).$$

There exists $b^{(2)}(\theta) \in W^{N,2}(\mathbb{S}^1)$ and $(\varphi(2), g^{(2)})$ solution of (3.1.1) *in the generalized wave coordinates $H_{b^{(2)}}$, such that, if we write $g^{(2)} = g_{b^2} + \widetilde{g}$, then $(\varphi(2), \widetilde{g}^{(2)})$ satisfies the same estimate as $(\varphi, \widetilde{g})$, and we have the estimates for $b^{(2)}$*

$$\begin{aligned}
&\left\|\partial_\theta^I\left(\Pi b^{(2)}(\theta) + \Pi\int_{\Sigma_{T,\theta}}(\partial_q\varphi)^2 r dq\right)\right\|_{L^2} \leq C\frac{\varepsilon^4}{\sqrt{T}}, \ for\ I \leq N-4,\\
&\|\partial_\theta^I b(\theta)\|_{L^2} \leq 2C_0^2\varepsilon^2, \ for\ I \leq N.
\end{aligned}$$

We may now prove Theorem 3.1.12.

Proof of Theorem 3.1.12. We may choose C_0 such that $C_0 \geq 2$, and B such that $B \geq 4C_0^2$. We take ε small enough so that

$$C\varepsilon^{\frac{1}{4}} \leq \frac{C_0}{2}, \quad C\sqrt{\varepsilon} \leq \rho, \quad C\varepsilon \leq \frac{B}{2}.$$

Then Propositions 3.4.4, 3.4.5, 3.4.6 imply that the bootstrap assumptions for $(\varphi, \widetilde{g})$ are true with the constant $2C_0$ replaced by $\frac{3C_0}{2}$. Moreover Proposition 3.4.7 yields the existence of $b^{(2)}$ and $\varphi^{(2)}, g^{(2)} = g_{b^{(2)}} + \widetilde{g}^{(2)}$ solution of (3.1.1), such that the bootstrap assumptions are satisfied by $(\varphi^{(2)}, \widetilde{g}^{(2)})$ with the constant $2C_0$ replaced by $\frac{3C_0}{2}$, and $b^{(2)}$ satisfy

$$\begin{aligned}
&\left\|\partial_\theta^I\left(\Pi b^{(2)}(\theta) + \Pi\int_{\Sigma_{T,\theta}}(\partial_q\varphi^{(2)})^2 r dq\right)\right\|_{L^2} \leq B\frac{\varepsilon^2}{2\sqrt{T}}, \ for\ I \leq N-4,\\
&\|\partial_\theta^I b(\theta)\|_{L^2} \leq \frac{B}{2}\varepsilon^2, \ for\ I \leq N.
\end{aligned}$$

This concludes the proof of Theorem 3.1.12. □

Let us note that the only place where we use the assumption $T \leq \exp\left(\frac{C}{\sqrt{\varepsilon}}\right)$ is in the proof of Proposition 3.4.7.

3.4.3 First consequences of the bootstrap assumptions

Thanks to the weighted Klainerman-Sobolev inequality the bootstrap assumptions immediately imply the following proposition.

Proposition 3.4.8. *We assume $I \le N-4$ we have the estimates, for $q<0$*

$$|\partial Z^I\varphi(t,x)| \lesssim \frac{\varepsilon}{\sqrt{1+|q|}\sqrt{1+s}}, \tag{3.4.28}$$
$$|\partial Z^I\widetilde{g}_3(t,x)| \lesssim \frac{\varepsilon(1+|q|)^{\mu}}{\sqrt{1+s}}, \tag{3.4.29}$$
$$|\partial Z^I h| \lesssim \frac{\varepsilon}{\sqrt{1+|q|}}, \tag{3.4.30}$$

and for $q>0$

$$|\partial Z^I\varphi(t,x)| \lesssim \frac{\varepsilon}{(1+|q|)^{\frac{3}{2}+\delta-\sigma}\sqrt{1+s}}, \tag{3.4.31}$$
$$|\partial Z^I\widetilde{g}_3(t,x)| \lesssim \frac{\varepsilon}{(1+|q|)^{\frac{3}{2}+\delta-\sigma}\sqrt{1+s}}, \tag{3.4.32}$$
$$|\partial Z^I h| \lesssim \frac{\varepsilon}{(1+|q|)^{2+\delta-\sigma}}. \tag{3.4.33}$$

Moreover, for $I \le N-11$ we have for $q<0$

$$|\partial Z^I\widetilde{g}_2(t,x)| \lesssim \frac{\varepsilon}{\sqrt{1+|q|}\sqrt{1+s}} \tag{3.4.34}$$

Thanks to Lemma 3.1.11 we deduce the following corollary.

Corollary 3.4.9. *We assume $I \le N-4$ we have the estimates, for $q<0$*

$$|Z^I\varphi(t,x)| \lesssim \frac{\varepsilon\sqrt{1+|q|}}{\sqrt{1+s}}, \tag{3.4.35}$$
$$|Z^I\widetilde{g}_3(t,x)| \lesssim \frac{\varepsilon(1+|q|)^{1+\mu}}{\sqrt{1+s}}, \tag{3.4.36}$$
$$|Z^I h| \lesssim \varepsilon\sqrt{1+|q|}. \tag{3.4.37}$$

and for $q>0$

$$|Z^I\varphi(t,x)| \lesssim \frac{\varepsilon}{(1+|q|)^{\frac{1}{2}+\delta-\sigma}\sqrt{1+s}}, \tag{3.4.38}$$
$$|Z^I\widetilde{g}_3(t,x)| \lesssim \frac{\varepsilon}{(1+|q|)^{\frac{1}{2}+\delta-\sigma}\sqrt{1+s}}, \tag{3.4.39}$$
$$|Z^I h| \lesssim \frac{\varepsilon}{(1+|q|)^{1+\delta-\sigma}}. \tag{3.4.40}$$

Moreover, for $I \le N-11$ we have for $q<0$

$$|Z^I\widetilde{g}_2(t,x)| \le \frac{\varepsilon\sqrt{1+|q|}}{\sqrt{1+s}}. \tag{3.4.41}$$

The following remark allow us to compare the different decompositions of the metric g.

Remark 3.4.10. *We have the following relations*

$$\widetilde{g}_{\mathcal{T}\mathcal{T}} = (\widetilde{g}_1)_{\mathcal{T}\mathcal{T}} = (\widetilde{g}_2)_{\mathcal{T}\mathcal{T}} = (\widetilde{g}_3)_{\mathcal{T}\mathcal{T}} = (\widetilde{g}_4)_{\mathcal{T}\mathcal{T}},$$
$$\widetilde{g}_{L\underline{L}} = (\widetilde{g}_1)_{L\underline{L}} = (\widetilde{g}_2)_{L\underline{L}} = (\widetilde{g}_3)_{L\underline{L}} = (\widetilde{g}_4)_{L\underline{L}},$$
$$\widetilde{g}_{U\underline{L}} = (\widetilde{g}_1)_{U\underline{L}} = (\widetilde{g}_2)_{U\underline{L}} = (\widetilde{g}_3)_{U\underline{L}}.$$

The following corollary allow us to estimate $\widetilde{g}$, independently of the chosen decomposition (3.4.6), (3.4.7), (3.4.8) or (3.4.9).

Corollary 3.4.11. *We have the following estimates*

$$|Z^I\widetilde{g}| \lesssim \frac{\varepsilon}{(1+|q|)^{\frac{1}{2}-\rho}}, \text{ for } I \leq N-14, \tag{3.4.42}$$
$$|Z^I\widetilde{g}| \lesssim \frac{\varepsilon}{(1+|q|)^{\frac{1}{2}-2\rho}}, \text{ for } I \leq N-12, \tag{3.4.43}$$
$$|Z^I\widetilde{g}| \lesssim \varepsilon, \quad |\partial Z^I\widetilde{g}| \lesssim \frac{\varepsilon}{1+|q|}, \text{ for } I \leq N-11, \tag{3.4.44}$$
$$|Z^I\widetilde{g}| \lesssim \varepsilon(1+|q|)^{\frac{1}{2}+\mu}, \quad |\partial Z^I\widetilde{g}| \lesssim \varepsilon(1+|q|)^{-\frac{1}{2}+\mu}, \text{ for } I \leq N-4. \tag{3.4.45}$$
$$\tag{3.4.46}$$

Moreover, for $q>0$ we have the following estimate

$$|Z^I\widetilde{g}| \lesssim \frac{\varepsilon}{(1+|q|)^{1+\delta-\sigma}}, \text{ for } I \leq N-4. \tag{3.4.47}$$

Proof. Estimate (3.4.42) is obtained by using the decomposition (3.4.6) and taking the maximum of the bounds given by (3.4.19) and (3.4.16). Estimate (3.4.43) is obtained by using the decomposition (3.4.6) and taking the maximum of the bounds given by (3.4.19) and (3.4.18). Estimate (3.4.44) is obtained by using the decomposition (3.4.7) and taking the maximum of the bounds given by (3.4.19), (3.4.21) and (3.4.40). Estimate (3.4.45) is obtained by using the decomposition (3.4.8) and taking the maximum of the bounds given by (3.4.37) and (3.4.36). Estimate (3.4.47) is obtained by using the decomposition (3.4.8) and taking the maximum of the bounds given by (3.4.40) and (3.4.39). □

The rest of the paper is as followed

- In Section 3.5, we use the wave coordinates condition to obtain better decay on the coefficients $g_{\mathcal{T}\mathcal{T}}$ of the metric. The strategy is similar to the one introduced in [40].
- In Section 3.6, we obtain the missing estimates for the angle and linear momentum, namely the three first Fourier coefficient of b which correspond to $b-\Pi b$, in order to get
$$\left| b(\theta) + \int_{\Sigma_{T,\theta}} (\partial_q\varphi(q, s=T,\theta))^2 r dq \right| \lesssim \frac{\varepsilon^2}{T^{\frac{1}{2}}},$$
by relying in particular on the constraint equations
- In Section 3.7, we improve the estimates for h_0, and show that it is indeed a good approximation for the coefficient $g_{L\underline{L}}$. We also obtain estimates for $\widetilde{h}$. We prove Proposition 3.4.4.
- In Section 3.8 we prove Proposition 3.4.5 thanks to the $L^\infty - L^\infty$ estimate.
- In Section 3.9 we derive a weighted energy estimate for an equation of the form $\Box_g u = f$, where g satisfies the bootstrap assumptions.
- In Section 3.10, we prove Proposition 3.4.6 thanks to the weighted energy estimate.
- In Section 3.11, we prove Proposition 3.4.7 by picking the right $\widetilde{b} = \Pi b$.

3.5 The wave coordinates condition

The wave coordinates condition yields better decay properties in t for some components of the metric. Since far from a conical neighborhoud of the light cone, we have $|q| \sim s$, this condition will only be relevant near the light cone. It is given by

$$H_b^\alpha = -\frac{1}{\sqrt{|det(g)|}}\partial_\mu(g^{\mu\alpha}\sqrt{|\det(g)|}).$$

Proposition 3.5.1. *We have the following estimate, in the region $\frac{t}{2} \leq r \leq 2t$,*

$$|\partial_q Z^I \widetilde{g}_{LL}| \lesssim \sum_{J\leq I}\left(|\bar{\partial}Z^J\widetilde{g}_{\underline{L}L}| + |\bar{\partial}Z^J\widetilde{g}_{\mathcal{T}\mathcal{T}}|\right) + \frac{1}{1+s}\sum_{J\leq I}\left(|Z^I\widetilde{g}_{L\underline{L}}| + |Z^I\widetilde{g}_{\mathcal{T}\mathcal{T}}|\right).$$

Proof. The wave coordinate condition implies

$$\begin{aligned}
-\underline{L}_\alpha H_b^\alpha = &\underline{L}_\alpha\left(\frac{1}{\sqrt{|\det(g)|}}\partial_\mu(g^{\mu\alpha}\sqrt{\det(g)})\right)\\
=&\frac{g^{\mu\alpha}}{\sqrt{\det(|g|)}}\underline{L}_\alpha\partial_\mu\sqrt{\det(g)} + \partial_\mu(\underline{L}_\alpha g^{\mu\alpha}) - g^{\mu\alpha}\partial_\mu(\underline{L}_\alpha)\\
=&\frac{g^{\underline{L}\mu}}{\sqrt{\det(|g|)}}\partial_\mu\sqrt{\det(g)} + \partial_\mu(g^{\underline{L}\mu}) - \frac{1}{r}g^{UU}\\
=&\frac{g^{\underline{L}\underline{L}}}{\sqrt{\det(|g|)}}\partial_{\underline{L}}\sqrt{\det(g)} + \frac{g^{\underline{L}\mathcal{T}}}{\sqrt{\det(|g|)}}\partial_{\mathcal{T}}\sqrt{\det(g)} + \partial_{\underline{L}}g^{\underline{L}\underline{L}} + \partial_U g^{\underline{L}U} + \partial_L g^{\underline{L}L}\\
&+ \frac{1}{r}g^{\underline{L}R} - \frac{1}{r}g^{UU},
\end{aligned}$$

where we have denoted by R the vector field ∂_r, and used the following calculations

$$\begin{aligned}
g^{\mu\alpha}\partial_\mu(\underline{L}_\alpha) =& -g^{\mu\alpha}\partial_\mu(R_\alpha)\\
=& -g^{11}\partial_1\cos(\theta) - g^{12}(\partial_2\cos(\theta) - \partial_1\sin(\theta)) - g^{22}\partial_2\sin(\theta)\\
=& -\frac{g^{UU}}{r},
\end{aligned}$$

$$\begin{aligned}
\partial_\mu g^{\underline{L}\mu} =&\partial_0 g^{\underline{L}0} + \partial_1 g^{\underline{L}1} + \partial_2 g^{\underline{L}2}\\
=&\partial_0 g^{\underline{L}0} + \partial_R g^{\underline{L}R} + \partial_U g^{\underline{L}U} + g^{\underline{L}R}(\partial_1\cos(\theta) + \partial_2\sin(\theta)) + g^{\underline{L}U}(-\partial_1\sin(\theta) + \partial_2\cos(\theta))\\
=&\partial_{\underline{L}}g^{\underline{L}\underline{L}} + \partial_U g^{\underline{L}U} + \partial_L g^{\underline{L}L} + \frac{g^{\underline{L}R}}{r}.
\end{aligned}$$

Consequently

$$\begin{aligned}
\partial_{\underline{L}}g^{\underline{L}\underline{L}} =& -\underline{L}_\alpha\left(\bar{H}_b^\alpha + F^\alpha\right) - \frac{g^{\underline{L}\underline{L}}}{\sqrt{\det(|g|)}}\partial_{\underline{L}}\sqrt{\det(g)} - \frac{g^{\underline{L}\mathcal{T}}}{\sqrt{\det(|g|)}}\partial_{\mathcal{T}}\sqrt{\det(g)}\\
&-\partial_U g^{\underline{L}U} - \partial_L g^{\underline{L}L} - \frac{1}{r}g^{\underline{L}R} - \frac{1}{r}g^{UU},
\end{aligned} \tag{3.5.1}$$

where we have used (3.1.9). Also we have

$$\det(g) = g_{LL}(g_{\underline{L}\underline{L}}g_{UU} - (g_{U\underline{L}})^2) - g_{L\underline{L}}(g_{L\underline{L}}g_{UU} - g_{LU}g_{\underline{L}U}) + g_{LU}(g_{\underline{L}L}g_{U\underline{L}} - g_{\underline{L}\underline{L}}g_{LU}).$$

Therefore

$$|\sqrt{\det(g)} - \sqrt{\det(g_b)}| \lesssim |\widetilde{g}_{\underline{L}L}| + |\widetilde{g}_{\mathcal{T}\mathcal{T}}|.$$

We can express

$$
\begin{aligned}
g^{\underline{L}\underline{L}} &= \frac{1}{\det(g)}(g_{LL}g_{UU} - (g_{UL})^2) = -\frac{1}{4}\widetilde{g}_{LL} + O(\widetilde{g}_{\mathcal{T}\mathcal{T}})O(g),\\
g^{\underline{L}U} &= \frac{1}{\det(g)}(g_{\underline{L}L}g_{LU} - g_{U\underline{L}}g_{LL}) = \frac{1}{2}g_{LU} + O(\widetilde{g}_{\mathcal{T}\mathcal{T}})O(g),\\
g^{\underline{L}L} &= \frac{1}{\det(g)}(g_{\underline{L}L}g_{UU} - g_{U\underline{L}}g_{UL}) = \frac{1}{4}(g_b)_{UU}g_{L\underline{L}} + O(\widetilde{g}_{\mathcal{T}\mathcal{T}}),
\end{aligned}
$$

where we have used the notation $O(g) = O(g-m)$ where m is the Minkowski metric. Since in (3.5.1), by definition of $\bar{H}^\alpha$ (see (3.1.10)) the terms involving only g_b compensate, we have

$$
|\partial_q \widetilde{g}_{LL}| \lesssim (|\bar{\partial}\widetilde{g}_{L\underline{L}}| + |\bar{\partial}\widetilde{g}_{\mathcal{T}\mathcal{T}}|) + \frac{1}{1+s}(|\widetilde{g}_{L\underline{L}}| + |\widetilde{g}_{\mathcal{T}\mathcal{T}}|) + s.t..
$$

where $s.t$ denotes similar terms (here these terms are quadratic terms with a better or similar decay), and we have used the fact that in the region $\frac{t}{2} \leq r \leq 2t$, we have $r \sim s$. Since $[Z, \partial_q] \sim \partial_q$ and $[Z, \bar{\partial}] \sim \bar{\partial}$ we have

$$
|\partial_q Z^I \widetilde{g}_{LL}| \lesssim \sum_{J \leq I-1} |Z^J \widetilde{g}_{LL}| + |\bar{\partial} Z^I \widetilde{g}_{L\underline{L}}| + |\bar{\partial} Z^I \widetilde{g}_{\mathcal{T}\mathcal{T}}| + \frac{1}{1+s}\sum_{J\leq I}(|Z^J\widetilde{g}_{L\underline{L}}| + |Z^J \widetilde{g}_{\mathcal{T}\mathcal{T}}|).
$$

This concludes the proof of Proposition 3.5.1. □

The other two contractions of the wave condition yield better decay on a conical neighbourhood of the light cone for $\widetilde{g}_{UL}$ and $\widetilde{g}_{UU}$.

Proposition 3.5.2. *We have the following property*

$$
\begin{aligned}
|\partial_q Z^I \widetilde{g}_{UL}| &\lesssim \sum_{J\leq I} |\bar{\partial} Z^J \widetilde{g}_{\mathcal{T}\mathcal{V}}| + \frac{1}{1+s}\sum_{J \leq I}|Z^J \widetilde{g}_{\mathcal{T}\mathcal{V}}|,\\
|\partial_q Z^I \widetilde{g}_{UU}| &\lesssim \sum_{J\leq I} |\bar{\partial} Z^J \widetilde{g}| + \frac{1}{1+s}\sum_{J \leq I}|Z^J \widetilde{g}|.
\end{aligned}
$$

Proof. To obtain the first estimate, we contract the wave coordinate condition with the vector field U.

$$
\begin{aligned}
-U_\alpha H_b^\alpha =& \frac{1}{\sqrt{|\det(g)|}} U_\alpha \partial_\mu (g^{\mu\alpha})\sqrt{\det(g)}\\
=& \frac{g^{\mu\alpha}}{\sqrt{|\det(g)|}} U_\alpha \partial_\mu \sqrt{|\det(g)|} + \partial_\mu(U_\alpha g^{\mu\alpha}) + g^{\mu\alpha}\partial_\mu(U_\alpha)\\
=& \frac{g^{U\mu}}{\sqrt{|\det(g)|}} \partial_\mu \sqrt{|\det(g)|} + \partial_\mu(g^{U\mu}) + \frac{1}{r}g^{UR}\\
=& \frac{g^{U\underline{L}}}{\sqrt{|\det(g)|}} \partial_{\underline{L}} \sqrt{|\det(g)|} + \frac{g^{U\mathcal{T}}}{\sqrt{|\det(g)|}} \partial_{\mathcal{T}} \sqrt{|\det(g)|} + \partial_{\underline{L}} g^{U\underline{L}} + \partial_U g^{UU} + \partial_L g^{UL} + \frac{1}{r}g^{UR}.
\end{aligned}
$$

Therefore

$$
\partial_{\underline{L}} g^{U\underline{L}} = -U_\alpha H_b^\alpha - \frac{g^{U\underline{L}}}{\sqrt{|\det(g)|}} \partial_{\underline{L}} \sqrt{|\det(g)|} - \frac{g^{U\mathcal{T}}}{\sqrt{|\det(g)|}} \partial_{\mathcal{T}} \sqrt{|\det(g)|} - \partial_U g^{UU} - \partial_L g^{UL} - \frac{1}{r}g^{UR}.
$$

and arguing as in Proposition 3.5.1 we infer

$$
|\partial_q \widetilde{g}_{UL}| \lesssim |\bar{\partial}\widetilde{g}_{\mathcal{T}\mathcal{V}}| + \frac{1}{1+s}|\widetilde{g}_{\mathcal{T}\mathcal{V}}| + s.t.
$$

Commuting with the vector fields Z as before, we obtain the desired estimate. To obtain the second one, we contract the wave coordinate condition with L

$$\begin{aligned} L_\alpha H_b^\alpha =& \frac{1}{\sqrt{|\det g|}} L_\alpha \partial_\mu (g^{\mu\alpha}) \sqrt{|\det(g)|}. \\ =& \frac{1}{\sqrt{|\det g|}} \partial_{\underline{L}} \left(\sqrt{|\det(g)|} g^{L\underline{L}} \right) + \frac{1}{\sqrt{|\det g|}} \partial_{\mathcal{T}} \left(\sqrt{|\det(g)|} g^{L\mathcal{T}} \right) - g^{\mu\alpha} \partial_\mu (L_\alpha). \end{aligned} \tag{3.5.2}$$

We note that

$$\begin{aligned} \sqrt{|\det(g)|} g^{L\underline{L}} =& \frac{1}{\sqrt{|\det(g)|}} (g_{L\underline{L}} g_{UU} - g_{U\underline{L}} g_{UL}) \\ =& \frac{g_{L\underline{L}} g_{UU}}{\sqrt{g_{L\underline{L}}^2 g_{UU} + O(\widetilde{g}_{\mathcal{T}\mathcal{T}}) O(g)}} + O(\widetilde{g}_{\mathcal{T}\mathcal{T}}) O(g) \\ =& \sqrt{g_{UU}} + + O(\widetilde{g}_{\mathcal{T}\mathcal{T}}) O(g). \end{aligned}$$

Therefore (3.5.2) yields

$$|\partial_q \widetilde{g}_{UU}| \lesssim |\bar{\partial} \widetilde{g}| + \frac{1}{1+s} |\widetilde{g}|.$$

We commute with the vector fields Z to conclude. □

Thanks to the bootstrap assumptions, we obtain the following corollary.

Corollary 3.5.3. *We have the estimates for $q < 0$*

$$|\partial Z^I \widetilde{g}_{UU}| \lesssim \frac{\varepsilon}{(1+s)^{\frac{3}{2}-\rho}}, \quad |\partial Z^I \widetilde{g}_{L\mathcal{T}}| \lesssim \frac{\varepsilon}{(1+s)(1+|q|)^{\frac{1}{2}-\rho}}, \quad \textit{for } I \leq N-15, \tag{3.5.3}$$

$$|\partial \widetilde{Z}^I g_{L\mathcal{T}}| \lesssim \frac{\varepsilon}{(1+s)^{\frac{3}{2}-2\rho}}, \quad |\partial Z^I \widetilde{g}_{UU}| \lesssim \frac{\varepsilon}{(1+s)(1+|q|)^{\frac{1}{2}-2\rho}}, \quad \textit{for } I \leq N-13, \tag{3.5.4}$$

$$|\partial Z^I \widetilde{g}_{L\mathcal{T}}| \lesssim \frac{\varepsilon\sqrt{1+|q|}}{(1+s)^{\frac{3}{2}}}, \quad |\partial Z^I \widetilde{g}_{UU}| \lesssim \frac{\varepsilon}{1+s}, \quad \textit{for } I \leq N-12, \tag{3.5.5}$$

$$|\partial Z^I \widetilde{g}_{L\mathcal{T}}| \lesssim \frac{\varepsilon(1+|q|)^{1+\mu}}{(1+s)^{\frac{3}{2}}}, \quad |\partial Z^I \widetilde{g}_{UU}| \lesssim \frac{\varepsilon(1+|q|)^{\frac{1}{2}+\mu}}{1+s}, \quad \textit{for } I \leq N-5, \tag{3.5.6}$$

and for $q > 0$

$$|\partial Z^I \widetilde{g}_{L\mathcal{T}}| \lesssim \frac{\varepsilon}{(1+|q|)^{\frac{1}{2}+\delta-\sigma}(1+s)^{\frac{3}{2}}}, \quad |\partial Z^I \widetilde{g}_{UU}| \lesssim \frac{\varepsilon}{(1+|q|)^{1+\delta-\sigma}(1+s)}, \quad \textit{for } I \leq N-5.$$

Proof. As mentioned in Remark 3.4.10, the metric coefficients $\widetilde{g}_{\mathcal{V}\mathcal{T}}$ do not depend on the choice of decomposition between (3.4.6), (3.4.7) and (3.4.8). Thanks to Proposition 3.5.1 and 3.5.2, and the fact that

$$|\bar{\partial} u| \leq \frac{1}{1+s} |Zu|,$$

we may write

$$|\partial Z^I \widetilde{g}_{L\mathcal{T}}| \lesssim \frac{1}{1+s} |Z^{I+1} \widetilde{g}_{\mathcal{T}\mathcal{V}}|. \tag{3.5.7}$$

The bootstrap assumptions (3.4.16) and (3.4.18) in the region $q < 0$ yield

$$\begin{aligned} |Z^J \widetilde{g}_{\mathcal{T}\mathcal{V}}| \lesssim& \frac{\varepsilon}{(1+s)^{\frac{1}{2}-\rho}}, \quad \textit{for } J \leq N-14, \\ |Z^J \widetilde{g}_{\mathcal{T}\mathcal{V}}| \lesssim& \frac{\varepsilon}{(1+s)^{\frac{1}{2}-2\rho}}, \quad \textit{for } J \leq N-12. \end{aligned}$$

Therefore we obtain, in view of (3.5.7)

$$|\partial Z^I \widetilde{g}_{L\mathcal{T}}| \lesssim \frac{\varepsilon}{(1+s)^{\frac{3}{2}-\rho}}, \quad for\ I \leq N-15,$$
$$|\partial Z^I \widetilde{g}_{L\mathcal{T}}| \lesssim \frac{\varepsilon}{(1+s)^{\frac{3}{2}-2\rho}} \quad for\ I \leq N-13.$$

Corollary 3.4.9 yields the following estimate for $q<0$

$$|Z^J \widetilde{g}_{\mathcal{T}\mathcal{V}}| \lesssim \frac{\varepsilon\sqrt{1+|q|}}{\sqrt{1+s}}, \quad for\ J \leq N-11,$$
$$|Z^J \widetilde{g}_{\mathcal{T}\mathcal{V}}| \lesssim \frac{\varepsilon(1+|q|)^{1+\mu}}{\sqrt{1+s}}, \quad for\ J \leq N-4.$$

Therefore we obtain in view of (3.5.7)

$$|\partial Z^I \widetilde{g}_{L\mathcal{T}}| \lesssim \frac{\varepsilon\sqrt{1+|q|}}{(1+s)^{\frac{3}{2}}} \quad for\ I \leq N-12,$$
$$|\partial Z^I \widetilde{g}_{L\mathcal{T}}| \lesssim \frac{\varepsilon(1+|q|)^{1+\mu}}{(1+s)^{\frac{3}{2}}} \quad for\ I \leq N-5.$$

For $q>0$ and $I \leq N-4$, we have in view of Corollary 3.4.9

$$|Z^I \widetilde{g}_{\mathcal{T}\mathcal{V}}| \lesssim \frac{\varepsilon}{\sqrt{1+s}(1+|q|)^{\frac{1}{2}+\delta-\sigma}}$$

which together with (3.5.7) yields

$$|\partial_q Z^I \widetilde{g}_{L\mathcal{T}}| \lesssim \frac{\varepsilon}{(1+|q|)^{\frac{1}{2}+\delta-\sigma}(1+s)^{\frac{3}{2}}} \quad for\ I \leq N-5.$$

We now estimate $Z^I \widetilde{g}_{UU}$. As for $Z^I \widetilde{g}_{L\mathcal{T}}$, Proposition 3.5.2 yields

$$|\partial Z^I \widetilde{g}_{UU}| \lesssim \frac{1}{1+s}|Z^{I+1}\widetilde{g}|.$$

Therefore, the estimates of Corollary 3.5.3 are a direct consequence of the estimates of Corollary 3.4.11. □

Thanks to Lemma 3.1.11, since $\delta-\sigma > \frac{1}{2}$ we obtain the following corollary

Corollary 3.5.4. *We have the estimates for $q<0$*

$$|Z^I \widetilde{g}_{L\mathcal{T}}| \lesssim \frac{\varepsilon(1+|q|)}{(1+s)^{\frac{3}{2}-\rho}}, \quad |Z^I \widetilde{g}_{UU}| \lesssim \frac{\varepsilon(1+|q|)^{\frac{1}{2}+\rho}}{1+s}, \quad for\ I \leq N-15, \tag{3.5.8}$$

$$|Z^I \widetilde{g}_{L\mathcal{T}}| \lesssim \frac{\varepsilon(1+|q|)}{(1+s)^{\frac{3}{2}-2\rho}} \quad |Z^I \widetilde{g}_{UU}| \lesssim \frac{\varepsilon(1+|q|)^{\frac{1}{2}+2\rho}}{1+s}, \quad for\ I \leq N-13, \tag{3.5.9}$$

$$|Z^I \widetilde{g}_{L\mathcal{T}}| \lesssim \frac{\varepsilon(1+|q|)^{\frac{3}{2}}}{(1+s)^{\frac{3}{2}}} \quad |Z^I \widetilde{g}_{UU}| \lesssim \frac{\varepsilon(1+|q|)}{1+s}, \quad for\ I \leq N-12, \tag{3.5.10}$$

$$|Z^I \widetilde{g}_{L\mathcal{T}}| \lesssim \frac{\varepsilon(1+|q|)^{2+\mu}}{(1+s)^{\frac{3}{2}}} \quad |Z^I \widetilde{g}_{UU}| \lesssim \frac{\varepsilon(1+|q|)^{\frac{3}{2}+\mu}}{1+s}, \quad for\ I \leq N-5, \tag{3.5.11}$$

and for $q>0$

$$|Z^I \widetilde{g}_{L\mathcal{T}}| \lesssim \frac{\varepsilon(1+|q|)^{\frac{1}{2}+\sigma-\delta}}{(1+s)^{\frac{3}{2}}}, \quad |Z^I \widetilde{g}_{UU}| \lesssim \frac{\varepsilon}{(1+s)(1+|q|)^{\delta-\sigma}}, \quad for\ I \leq N-5. \tag{3.5.12}$$

3.6 Angle and linear momentum

We call angle and linear momentum the three first coefficients of b, b_0, b_1, b_2. These coefficients can not be prescribed arbitrarily, they are given by the resolution of the constraint equations (see Theorem 3.1.3). We need b to satisfy

$$\left\| \partial_\theta^I \left(b(\theta) + \int_{\Sigma_{T,\theta}} (\partial_q \varphi)^2 r dq \right) \right\|_{L^2} \lesssim \frac{\varepsilon^2}{\sqrt{T}},\ for\ I \leq N-4. \tag{3.6.1}$$

This is used crucially to estimate h_0 in the proof of Proposition 3.7.2. The heuristic of it is discussed in Section 3.3.2 (see (3.3.15)). The estimate (3.6.1) is satisfied with b replaced by Πb thanks to the bootstrap assumption (3.4.4). For the angle and linear momentum, this is the object of the following proposition, which says that the relations of Theorem 3.12.1 are asymptotically conserved by the flow of the Einstein equations.

Proposition 3.6.1. *We have*

$$\left| \int b(\theta) d\theta + \frac{1}{2} \int_{\mathbb{R}^2} \left((\partial_t \varphi)^2 + |\nabla \varphi|^2 \right)(t,x) dx \right| \lesssim \frac{\varepsilon^2}{\sqrt{1+t}},$$
$$\left| \int b(\theta) \cos(\theta) d\theta - \int_{\mathbb{R}^2} \left(\partial_t \varphi \partial_1 \varphi \right)(t,x) dx \right| \lesssim \frac{\varepsilon^2}{\sqrt{1+t}},$$
$$\left| \int b(\theta) \sin(\theta) d\theta - \int_{\mathbb{R}^2} \left(\partial_t \varphi \partial_2 \varphi \right)(t,x) dx \right| \lesssim \frac{\varepsilon^2}{\sqrt{1+t}}.$$

To prove this proposition, we need the following lemma.

Lemma 3.6.2. *The equation for $g_{\mu\nu}$ can be written under the form*

$$\Box \widetilde{g}_{\mu\nu} = -2\partial_\mu \varphi \partial_\nu \varphi - 2b(\theta) \frac{\partial_q^2(\chi(q)q)}{r} M_{\mu\nu} + O\left(\frac{\varepsilon^2}{(1+t)^{\frac{3}{2}}(1+|q|)^{\frac{3}{2}-2\rho}} \right), \tag{3.6.2}$$

where the tensor $M_{\mu\nu}$ corresponds to dq^2.

Proof of Lemma 3.6.2. We recall the quasilinear equation for $\widetilde{g}_{\mu\nu}$ (see (3.2.4))

$$g^{\alpha\beta}\partial_\alpha \partial_\beta \widetilde{g}_{\mu\nu} - H_b^\rho \partial_\rho \widetilde{g}_{\mu\nu} = -2\partial_\mu \varphi \partial_\nu \varphi + 2(R_b)_{\mu\nu} + P_{\mu\nu}(\partial \widetilde{g}, \partial \widetilde{g}) + \widetilde{P}_{\mu\nu}(\widetilde{g}, g_b).$$

The worst term in

$$g^{\alpha\beta}\partial_\alpha \partial_\beta \widetilde{g}_{\mu\nu} - \Box \widetilde{g}_{\mu\nu}$$

is, according to Remark 3.2.2,

$$g_{LL} \partial_q^2 \widetilde{g}_{\mu\nu}.$$

We distinguish two kinds of contributions :

$$g_{LL} \partial_q^2 \widetilde{g}_1 \quad and \quad g_{LL} \partial_q^2 h_0.$$

To estimate the first term, we use (3.5.8) of Corollary 3.5.4, which gives

$$|g_{LL}| \lesssim \frac{\varepsilon(1+|q|)}{(1+s)^{\frac{3}{2}-\rho}}.$$

We estimate then

$$|\partial_q^2 \widetilde{g}_1| \leq \frac{1}{(1+|q|)^2} \sum_{I \leq 2} |Z^I \widetilde{g}_1|,$$

and we use the bootstrap assumption (3.4.16) for $I \leq N-14$

$$|Z^I \widetilde{g}_1| \leq \frac{\varepsilon}{(1+s)^{\frac{1}{2}-\rho}},$$

to obtain

$$|g_{LL}\partial_q^2 \widetilde{g}_1| \lesssim \frac{\varepsilon^2}{(1+s)^{2-2\rho}(1+|q|)}. \tag{3.6.3}$$

We now estimate the second term. To estimate $\partial_q^2 h_0$, we recall (3.4.19) for $I \leq N-6$

$$|Z^I h_0| \lesssim \frac{\varepsilon}{\sqrt{1+s}} + \frac{\varepsilon}{(1+|q|)^{1-4\rho}}.$$

Consequently

$$|\partial_q^2 h_0| \lesssim \frac{\varepsilon}{(1+|q|)^2\sqrt{1+s}} + \frac{\varepsilon}{(1+|q|)^{3-4\rho}}.$$

The first contribution can be estimated like 3.6.3. To tackle the second contribution we need to use the estimate for g_{LL} which gives the most decay in s : we use (3.5.10) of Corollary 3.5.4, which yields

$$|g_{LL}| \lesssim \frac{\varepsilon(1+|q|)^{\frac{3}{2}}}{(1+s)^{\frac{3}{2}}}.$$

This, together with the estimate (3.6.3), yields

$$|g_{LL}\partial_q^2 h_0| \lesssim \frac{\varepsilon^2}{(1+s)^{\frac{3}{2}}(1+|q|)^{\frac{3}{2}-4\rho}} + \frac{\varepsilon^2}{(1+s)^{2-2\rho}(1+|q|)}. \tag{3.6.4}$$

The semi linear terms $P_{\mu\nu}(\partial\widetilde{g},\partial\widetilde{g})$ are estimated similarly. We now turn to the crossed terms. Thanks to Section 3.2.5, the worst contribution is (3.2.11), which gives a contribution of the form $\frac{\varepsilon}{r}\partial\widetilde{g}_{L\underline{L}}$ in the region $q>0$. We estimate thanks to (3.4.32) of Corollary 3.4.9 in the region $q>0$

$$|\partial\widetilde{g}_{L\underline{L}}| \lesssim \frac{\varepsilon}{(1+s)^{\frac{1}{2}}(1+|q|)^{\frac{3}{2}+\delta-\sigma}}.$$

Therefore we obtain

$$\left|\mathbb{1}_{q>0}\frac{\varepsilon}{r}\partial\widetilde{g}_{\underline{L}L}\right| \lesssim \frac{\varepsilon}{(1+s)^2(1+|q|)^{\frac{3}{2}+\delta-\sigma}}. \tag{3.6.5}$$

We now estimate $(R_b)_{\mu\nu}$. Thanks to (3.1.7) and (3.1.8), we may write

$$(R_b)_{\mu\nu} = -\frac{b(\theta)\partial_q^2(q\chi(q))}{r}M_{\mu\nu} + O\left(\frac{\mathbb{1}_{1\leq q\leq 2}\varepsilon^2}{(1+r)^2}\right). \tag{3.6.6}$$

Thanks to (3.6.3), (3.6.4), (3.6.5) and (3.6.6) we conclude the proof of Lemma 3.6.2. □

Proof of Proposition 3.6.1. We want to integrate equation (3.6.2) for $(\mu,\nu)=0,0$ over the space-like hypersurfaces of t constant. To deal with the term $\partial_t^2 g_{00}$, we use the wave coordinate condition

$$g^{\alpha\beta}\partial_\beta g_{\alpha 0} = \frac{1}{2}g^{\alpha\beta}\partial_t g_{\alpha\beta} + (H_b)_0.$$

We can rewrite it, by definition of $(H_b)_0$

$$(g^{\alpha\beta}-(g_b)^{\alpha\beta})\partial_\beta g_{\alpha 0} + g_b^{\alpha\beta}(\partial_\beta g_{\alpha 0} - \partial_\beta (g_b)_{\alpha 0}) = \frac{1}{2}(g^{\alpha\beta}-g_b^{\alpha\beta})\partial_t g_{\alpha\beta} + \frac{1}{2}g_b^{\alpha\beta}(\partial_t g_{\alpha\beta} - \partial_t (g_b)_{\alpha\beta}) + F_0.$$

By definition, F contains only terms of the form $\widetilde{g}\partial_U g_b$, so we can estimate

$$|ZF| \lesssim \frac{\varepsilon \mathbb{1}_{q>0}(1+|q|)}{r^2}|Z\widetilde{g}| \lesssim \frac{\varepsilon^2}{(1+s)^2(1+|q|)^{\delta-\sigma}}, \tag{3.6.7}$$

where we have used (3.4.47) to estimate $|Z\widetilde{g}|$. We note

$$m^{\alpha\beta}\partial_\beta \widetilde{g}_{\alpha 0} - \frac{1}{2}m^{\alpha\beta}\partial_t \widetilde{g}_{\alpha\beta} = \frac{1}{2}(-\partial_t \widetilde{g}_{00} - \partial_t \widetilde{g}_{11} - \partial_t \widetilde{g}_{22}) + \partial_1 \widetilde{g}_{01} + \partial_2 \widetilde{g}_{02},$$

and we estimate

$$\begin{aligned}
(g^{\alpha\beta} - (g_b)^{\alpha\beta})\partial_\beta g_{\alpha 0} &= (g^{L\underline{L}} - m^{L\underline{L}})\partial_{\underline{L}} g_{L0} + f_1,\\
\frac{1}{2}(g^{\alpha\beta} - g_b^{\alpha\beta})\partial_t g_{\alpha\beta} &= (g^{L\underline{L}} - m^{L\underline{L}})\partial_t \widetilde{g}_{L\underline{L}} + f_2,\\
(m^{\alpha\beta} - g_b^{\alpha\beta})\partial_\beta \widetilde{g}_{\alpha 0} &= f_4\\
(m^{\alpha\beta} - g_b^{\alpha\beta})\partial_t \widetilde{g}_{\alpha\beta} &= f_5,
\end{aligned}$$

where the f_i contain terms of the form

$$\widetilde{g}_{LL}\partial \widetilde{g}_{\mathcal{V}\mathcal{V}}, \quad \widetilde{g}_{\mathcal{V}\mathcal{V}}\partial_{\mathcal{T}} g_{\mathcal{T}\mathcal{V}}, \quad \frac{b\chi(q)}{r}\partial_U \widetilde{g}_{U\mathcal{V}}, \quad \dots$$

They satisfy the following estimate

$$|Zf_i| \lesssim \frac{\varepsilon^2}{(1+s)^{\frac{3}{2}}(1+|q|)^{\frac{1}{2}-2\rho}}. \tag{3.6.8}$$

We note $2\partial_t \widetilde{g}_{L\underline{L}} = \partial_L \widetilde{g}_{L\underline{L}} + \partial_{\underline{L}} \widetilde{g}_{L\underline{L}}$ and $2g_{L0} = g_{L\underline{L}} + g_{LL}$. Consequently

$$(g^{L\underline{L}} - m^{L\underline{L}})(\partial_{\underline{L}} g_{L0} - \partial_t \widetilde{g}_{L\underline{L}}) = O\left(\widetilde{g}_{L\underline{L}}\partial_L \widetilde{g}_{L\underline{L}} + \widetilde{g}_{L\underline{L}}\partial_{\underline{L}} \widetilde{g}_{LL}\right)$$

satisfies the same estimate (3.6.8) than the f_i. Therefore the wave coordinate condition gives

$$\frac{1}{2}(-\partial_t \widetilde{g}_{00} - \partial_t \widetilde{g}_{11} - \partial_t \widetilde{g}_{22}) + \partial_1 \widetilde{g}_{01} + \partial_2 \widetilde{g}_{02} = f_5$$

where f_5 satisfies (3.6.8). Therefore, differentiating this equation with respect to t, and using (3.6.2) for $(\mu,\nu) = (0,0),(1,1),(2,2)$ we obtain

$$\begin{aligned}
&\Delta \widetilde{g}_{00} + \Delta \widetilde{g}_{11} + \Delta \widetilde{g}_{22} - 2\partial_1 \partial_t \widetilde{g}_{01} - 2\partial_2 \partial_t \widetilde{g}_{02}\\
=& -2((\partial_0 \varphi)^2 + (\partial_1 \varphi)^2 + (\partial_2 \varphi)^2) - 4b(\theta)\frac{\partial_q^2(\chi(q)q)}{r} + O\left(\frac{\varepsilon^2}{(1+s)^{\frac{3}{2}}(1+|q|)^{\frac{3}{2}-2\rho}}\right).
\end{aligned}$$

Integrating on the space-like hypersurface t constant we obtain, since $\int_0^\infty \partial^2(q\chi(q))dr = 1$,

$$-\frac{1}{2}\int (\partial_t \varphi)^2 + |\nabla \varphi|^2 = \int b(\theta)d\theta + O\left(\frac{\varepsilon^2}{\sqrt{1+t}}\right). \tag{3.6.9}$$

To obtain the next relation we do the same reasoning but with (3.6.2) for $(\mu,\nu) = (0,1)$ and $(\mu,\nu) = (0,2)$. We only detail the case $(\mu,\nu) = (0,1)$ as the other one is treated in the same way. Recall the wave coordinates condition

$$g^{\alpha\beta}\partial_\beta g_{\alpha 1} = \frac{1}{2}g^{\alpha\beta}\partial_1 g_{\alpha\beta} + (H_b)_1.$$

We can rewrite it, by definition of $(H_b)_1$

$$(g^{\alpha\beta}-(g_b)^{\alpha\beta})\partial_\beta g_{\alpha 1}+g_b^{\alpha\beta}(\partial_\beta g_{\alpha 1}-\partial_\beta (g_b)_{\alpha 1})=\frac{1}{2}(g^{\alpha\beta}-g_b^{\alpha\beta})\partial_1 g_{\alpha\beta}+\frac{1}{2}g_b^{\alpha\beta}(\partial_1 g_{\alpha\beta}-\partial_1(g_b)_{\alpha\beta})+F_1$$

We note

$$m^{\alpha\beta}\partial_\beta\widetilde{g}_{\alpha 1}-\frac{1}{2}m^{\alpha\beta}\partial_1\widetilde{g}_{\alpha\beta}=-\partial_t\widetilde{g}_{01}+\partial_1\widetilde{g}_{11}+\partial_2\widetilde{g}_{12}-\frac{1}{2}m^{\alpha\beta}\partial_1\widetilde{g}_{\alpha\beta}$$

and we estimate

$$\begin{aligned}
&(g^{\alpha\beta}-(g_b)^{\alpha\beta})\partial_\beta g_{\alpha 1}=(g^{L\underline{L}}-m^{L\underline{L}})\partial_{\underline{L}}g_{L1}+f_6,\\
&\frac{1}{2}(g^{\alpha\beta}-g_b^{\alpha\beta})\partial_1 g_{\alpha\beta}=(g^{L\underline{L}}-m^{L\underline{L}})\partial_1\widetilde{g}_{L\underline{L}}+f_7,\\
&(m^{\alpha\beta}-g_b^{\alpha\beta})\partial_\beta\widetilde{g}_{\alpha 1}=f_8,\\
&(m^{\alpha\beta}-g_b^{\alpha\beta})\partial_1\widetilde{g}_{\alpha\beta}=f_9,
\end{aligned}$$

where the quantities f_i satisfy (3.6.8). We note $2\partial_1\widetilde{g}_{L\underline{L}}=-\cos(\theta)\partial_{\underline{L}}\widetilde{g}_{L\underline{L}}+\bar{\partial}\widetilde{g}_{L\underline{L}}$ and $2\partial_{\underline{L}}\widetilde{g}_{L1}=-\partial_{\underline{L}}(\cos(\theta)g_{L\underline{L}})+g_{L\mathcal{T}}$. Therefore we obtain

$$-\partial_t\widetilde{g}_{01}+\partial_1\widetilde{g}_{11}+\partial_2\widetilde{g}_{12}-\frac{1}{2}m^{\alpha\beta}\partial_1\widetilde{g}_{\alpha\beta}=f_{10},$$

where f_{10} satisfies (3.6.8). Differentiating with respect to t and using (3.6.2) for $(\mu,\nu)=(0,1)$ we obtain

$$\begin{aligned}
&\Delta\widetilde{g}_{01}+\partial_1\partial_t\widetilde{g}_{11}+\partial_2\partial_t\widetilde{g}_{12}-\frac{1}{2}m^{\alpha\beta}\partial_1\partial_t\widetilde{g}_{\alpha\beta}\\
=&-2\partial_t\varphi\partial_1\varphi+2b(\theta)\cos(\theta)\frac{\partial_q^2(\chi(q)q)}{r}+O\left(\frac{\varepsilon^2}{(1+s)^{\frac{3}{2}}(1+|q|)^{\frac{3}{2\rho}}}\right).
\end{aligned}$$

Integrating on the space-like hypersurface t constant we obtain

$$\int\partial_t\varphi\partial_1\varphi=\int b(\theta)\cos(\theta)d\theta+O\left(\frac{\varepsilon^2}{\sqrt{1+t}}\right),\tag{3.6.10}$$

and similarly

$$\int\partial_t\varphi\partial_2\varphi=\int b(\theta)\sin(\theta)d\theta+O\left(\frac{\varepsilon^2}{\sqrt{1+t}}\right).\tag{3.6.11}$$

Estimates (3.6.9), (3.6.10) and (3.6.11) conclude the proof of Proposition 3.6.1 □

Corollary 3.6.3. *We have the estimates*

$$\begin{aligned}
\left|\int b(\theta)d\theta+\int_{\Sigma_T}(\partial_q\varphi)^2 rdrd\theta\right|&\lesssim\frac{\varepsilon^2}{\sqrt{T}}\\
\left|\int b(\theta)\cos(\theta)d\theta+\int_{\Sigma_T}\cos(\theta)(\partial_q\varphi)^2 rdrd\theta\right|&\lesssim\frac{\varepsilon^2}{\sqrt{T}}\\
\left|\int b(\theta)\sin(\theta)d\theta+\int_{\Sigma_T}\sin(\theta)(\partial_q\varphi)^2 rdrd\theta\right|&\lesssim\frac{\varepsilon^2}{\sqrt{T}}
\end{aligned}$$

Proof. We may write

$$\begin{aligned}
&\partial_t\varphi=-\partial_q\varphi+\partial_s\varphi,\\
&\partial_1\varphi=\cos(\theta)\partial_q\varphi+\cos(\theta)\partial_s\varphi-\sin(\theta)\partial_U\varphi\\
&\partial_1\varphi=\sin(\theta)\partial_q\varphi+\sin(\theta)\partial_s\varphi+\cos(\theta)\partial_U\varphi.
\end{aligned}$$

Moreover, thanks to the bootstrap assumption (3.4.15)

$$|\partial\varphi\bar{\partial}\varphi| \lesssim \frac{1}{(1+|q|)(1+s)}|Z\varphi|^2 \lesssim \frac{\varepsilon^2}{(1+s)^2(1+|q|)^{2-8\rho}},$$

and consequently

$$\left|\int \left(\partial\varphi\bar{\partial}\varphi\right)(t,x)dx\right| \lesssim \frac{\varepsilon^2}{1+t}.$$

Therefore

$$\left|2\int_{\Sigma_T}(\partial_q\varphi)^2dx - \int_{\Sigma_T}((\partial_t\varphi)^2+|\nabla\varphi|^2)dx\right| \lesssim \frac{\varepsilon^2}{1+T},$$
$$\left|\int_{\Sigma_T}\cos(\theta)(\partial_q\varphi)^2dx + \int_{\Sigma_T}\partial_t\varphi\partial_1\varphi dx\right| \lesssim \frac{\varepsilon^2}{1+T},$$
$$\left|\int_{\Sigma_T}\sin(\theta)(\partial_q\varphi)^2dx + \int_{\Sigma_T}\partial_t\varphi\partial_2\varphi dx\right| \lesssim \frac{\varepsilon^2}{1+T}.$$

This concludes the proof of Corollary 3.6.3. □

Corollary 3.6.3 and the bootstrap assumption 3.4.4 directly imply the following corollary.

Corollary 3.6.4. *We have, for $I \leq N-4$*

$$\left|\partial_\theta^I\left(b(\theta)+\int_{\Sigma_{T,\theta}}(\partial_q\varphi)^2rdq\right)\right| \lesssim \frac{\varepsilon^2}{\sqrt{T}}.$$

3.7 The transport equation (3.4.10)

In this section we will estimate h_0, $\Box h_0$ and $\widetilde{h}$.

3.7.1 Estimations on h_0

We recall the equation (3.4.10)

$$\begin{cases} \partial_q h_0 = -2r(\partial_q\varphi)^2 - 2b(\theta)\partial_q^2(\chi(q)q), \\ h_0|_{t=0} = 0. \end{cases}$$

The solution of this equation is

$$h_0(s,Q,\theta) = \int_s^Q \left(-2(\partial_q\varphi)^2 - 2\frac{b(\theta)\partial_q^2(q\chi(q))}{r}\right)rdq. \tag{3.7.1}$$

All the estimates we will perform in this section take place in the region $r > \frac{t}{2}$ since we will always apply the cut-off function $\Upsilon\left(\frac{r}{t}\right)$ to h_0.

Proposition 3.7.1. *In the region $r > \frac{t}{2}$ we have the estimates on h_0, for $q<0$*

$$|\partial_s h_0| \lesssim \frac{\varepsilon^2}{(1+s)^{\frac{3}{2}}}, \quad |h_0| \lesssim \frac{\varepsilon^2}{\sqrt{1+s}} + \frac{\varepsilon^2}{(1+|q|)^{2-8\rho}}$$

and for $q>0$

$$|\partial_s h_0| \lesssim \frac{\varepsilon^2}{(1+s)^{\frac{3}{2}}(1+|q|)^{\frac{3}{2}+2(\delta-\sigma)}}, \quad |h_0| \lesssim \frac{\varepsilon^2}{(1+|q|)^{2+2(\delta-\sigma)}}.$$

Proof. We write the wave operator in coordinates (s,q,θ)

$$\Box = 4\partial_s\partial_q + \frac{1}{r}(\partial_s+\partial_q) + \frac{1}{r^2}\partial_\theta^2. \tag{3.7.2}$$

We calculate

$$\partial_s\partial_q h_0 = \partial_s(-2r(\partial_q\varphi)^2) = -r\partial_q\varphi\left(4\partial_s\partial_q\varphi + \frac{1}{r}\partial_q\varphi\right) = -r\partial_q\varphi\left(\Box\varphi - \frac{1}{r}\partial_s\varphi - \frac{1}{r^2}\partial_\theta^2\varphi\right), \tag{3.7.3}$$

where we have used

$$\partial_s(-2b(\theta)\partial_q^2(q\chi(q))) = 0.$$

Therefore we have

$$\partial_s h_0 = \int_s^Q \left(-\Box\varphi + \frac{1}{r}\partial_s\varphi + \frac{1}{r^2}\partial_\theta^2\varphi\right)\partial_q\varphi r dq + O\left(\frac{\varepsilon^2}{(1+s)^{3+2\delta}}\right), \tag{3.7.4}$$

where we have used

$$\partial_s h_0|_{t=0} = -\partial_q h_0|_{t=0} = \left(2r(\partial_q\varphi)^2 + 2b(\theta)\partial_q^2(\chi(q)q)\right)|_{t=0} = O\left(\frac{\varepsilon^2}{(1+s)^{3+2\delta}}\right).$$

The bootstrap assumption (3.4.15) gives

$$\left|\frac{1}{r}\partial_s\varphi\right| + \left|\frac{1}{r^2}\partial_\theta^2\varphi\right| \lesssim \frac{1}{(1+s)^2}|Z^2\varphi| \lesssim \frac{\varepsilon}{(1+s)^{\frac{5}{2}}(1+|q|)^{\frac{1}{2}-4\rho}},$$

and

$$|\partial_q\varphi| \lesssim \frac{1}{1+|q|}|Z\varphi| \lesssim \frac{\varepsilon}{(1+s)^{\frac{1}{2}}(1+|q|)^{\frac{3}{2}-4\rho}}.$$

Therefore

$$\left|\left(\frac{1}{r}\partial_s\varphi + \frac{1}{r^2}\partial_\theta^2\varphi\right)\partial_q\varphi r\right| \lesssim \frac{\varepsilon^2}{(1+s)^2(1+|q|)^{2-8\rho}}. \tag{3.7.5}$$

To estimate $\Box\varphi$ we write $\Box\varphi = (\Box - \Box_g)\varphi$. Thanks to Remark 3.2.2, in the region $q<0$ it is sufficient to estimate $g_{LL}\partial_q^2\varphi$. We start with the region $q<0$. To obtain all the possible decay in s, we use the estimate (3.5.10) of Corollary 3.5.4 for $I \le N-11$, which gives, for $q<0$

$$|g_{LL}| \lesssim \frac{\varepsilon(1+|q|)^{\frac{3}{2}}}{(1+s)^{\frac{3}{2}}}.$$

The bootstrap assumption (3.4.15) imply

$$|\partial_q^2\varphi| \lesssim \frac{\varepsilon}{(1+|q|)^{\frac{5}{2}-4\rho}\sqrt{1+s}},$$

therefore

$$|g_{LL}\partial_q^2\varphi\partial_q\varphi| \lesssim \frac{\varepsilon^3(1+|q|)^{\frac{3}{2}}}{(1+s)^{\frac{5}{2}}(1+|q|)^{4-8\rho}},$$

and we obtain

$$|(\Box\varphi)\partial_q\varphi r| \lesssim \frac{\varepsilon^3}{(1+s)^{\frac{3}{2}}(1+|q|)^{\frac{5}{2}-8\rho}}. \tag{3.7.6}$$

Thanks to (3.7.5) and (3.7.6), in the region $q<0$ we have

$$\left|\left(-\Box\varphi + \frac{1}{r}\partial_s\varphi + \frac{1}{r^2}\partial_\theta^2\varphi\right)\partial_q\varphi r\right| \lesssim \frac{\varepsilon^3}{(1+s)^{\frac{3}{2}}(1+|q|)^{\frac{5}{2}-8\rho}}. \tag{3.7.7}$$

We now estimate the integrand in the region $q > 0$. Estimate (3.4.38) yields, for $q > 0$ and $I \leq N-3$

$$|Z^I\varphi| \lesssim \frac{\varepsilon}{\sqrt{1+s}(1+|q|)^{\frac{1}{2}+\delta-\sigma}},$$

and estimate (3.5.12) yields for $q > 0$

$$|g_{LL}| \lesssim \frac{(1+|q|)^{\frac{1}{2}+\sigma-\delta}}{(1+s)^{\frac{3}{2}}}.$$

In the region $q > 0$, $\Box\varphi - \Box_g\varphi$ contains also terms of the form $\frac{\varepsilon\chi(q)}{r}\bar{\partial}\varphi$ (see (3.2.13) in the discussion of Section 3.2.5). We can neglect them since we already take into account terms of the form $\frac{1}{r}\partial_s\varphi + \frac{1}{r^2}\partial_\theta^2\varphi$ in (3.7.3). Consequently for $q > 0$

$$\left|\left(-\Box\varphi + \frac{1}{r}\partial_s\varphi + \frac{1}{r^2}\partial_\theta^2\varphi\right)\partial_q\varphi r\right| \lesssim \frac{\varepsilon^2}{(1+s)^{\frac{3}{2}}(1+|q|)^{\frac{5}{2}+2\delta-2\sigma}}. \tag{3.7.8}$$

Therefore, (3.7.4) and (3.7.8) yield for $q > 0$

$$|\partial_s h_0| \lesssim \frac{\varepsilon^2}{(1+s)^{\frac{3}{2}}(1+|q|)^{\frac{3}{2}+2(\delta-\sigma)}}, \tag{3.7.9}$$

and (3.7.4), (3.7.7) and (3.7.8) yield for $q < 0$, since $\frac{1}{(1+|q|)^{\frac{5}{2}-8\rho}}$ is integrable,

$$|\partial_s h_0| \lesssim \frac{\varepsilon^2}{(1+s)^{\frac{3}{2}}}. \tag{3.7.10}$$

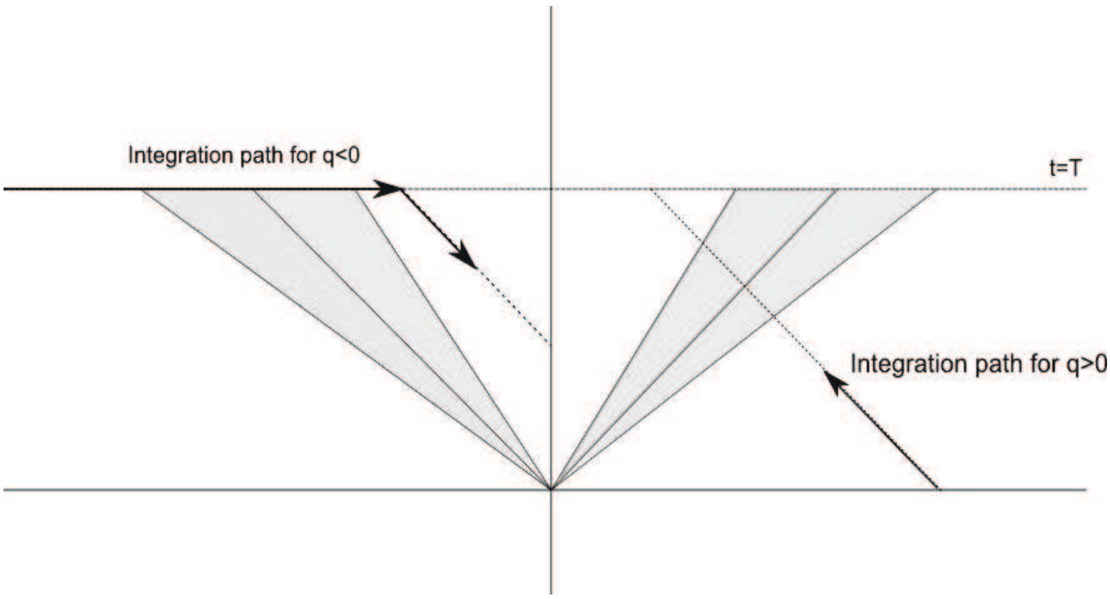

Figure 3.1: Integration of h_0

Thanks to Corollary 3.6.4 we have

$$\left|b(\theta) + \int_{\Sigma_{T,\theta}} (\partial_q\varphi)^2 r dr\right| \lesssim \frac{\varepsilon^2}{T^{\frac{1}{2}}}.$$

Moreover $\partial_r h_0 = \partial_q h_0 + \partial_s h_0$ and therefore (3.7.10) and (3.7.9) yield

$$\partial_r h_0 = -2r(\partial_q\varphi)^2 - 2b(\theta)\partial_r^2(\chi(q)q) + O\left(\frac{\varepsilon^2}{(1+s)^{\frac{3}{2}}}\right). \tag{3.7.11}$$

Therefore, on the line $t = T$, with fixed θ we obtain the following estimate for h_0 in the region $r < t$ by integrating (3.7.11)

$$\begin{aligned} h_0(T,R,\theta) = & -\int_R^\infty \left(-2r(\partial_q\varphi)^2 + O\left(\frac{\varepsilon^2}{(r+T)^{\frac{3}{2}}}\right)\right) + 2b(\theta) \\ = & \int_0^R 2r(\partial_q\varphi)^2 dr + O\left(\frac{\varepsilon^2}{\sqrt{1+T}}\right) \\ = & O\left(\frac{\varepsilon^2}{(1+T)^{\frac{1}{2}}}\right) + O\left(\frac{\varepsilon^2}{(1+q)^{2-8\rho}}\right). \end{aligned}$$

To estimate h_0 elsewhere in the region $r < t$, we can integrate the estimate (3.7.10), at fixed q, as shown in left of the figure 3.7.1. To estimate h_0 in the region $r > t$ we integrate the transport equation from $t = 0$, as shown in the right of the figure 3.7.1 : we rely on formula (3.7.1) and the estimate for $q > 0$

$$|\partial_q\varphi| \lesssim \frac{\varepsilon}{\sqrt{1+s}(1+|q|)^{\frac{3}{2}+\delta-\sigma}}.$$

We obtain

$$\begin{aligned} h_0 &= O\left(\frac{\varepsilon^2}{(1+q)^{2+2(\delta-\sigma)}}\right),\ q > 0, \\ h_0 &= O\left(\frac{\varepsilon^2}{(1+s)^{\frac{1}{2}}}\right) + O\left(\frac{\varepsilon^2}{(1+q)^{2-8\rho}}\right),\ q < 0. \end{aligned}$$

□

Next we derive an estimate for $Z^I h_0$.

Proposition 3.7.2. *Let $I \leq N-5$. We have the estimate for $q < 0$*

$$|Z^I h_0| \lesssim \frac{\varepsilon^2}{\sqrt{1+s}} + \frac{\varepsilon^2}{(1+|q|)^{1-4\rho}}, \quad |\partial_s Z^I h_0| \lesssim \frac{\varepsilon^2}{(1+s)^{\frac{3}{2}}}, \quad |\partial_q Z^I h_0| \lesssim \frac{\varepsilon^2}{(1+|q|)^{2-4\rho}},$$

and for $q > 0$

$$|Z^I h_0| \lesssim \frac{\varepsilon^2}{(1+|q|)^{2+2(\delta-\sigma)}}, \quad |\partial_s Z^I h_0| \lesssim \frac{\varepsilon^2}{(1+s)^{\frac{3}{2}}(1+|q|)^{1+2(\delta-\sigma)}}.$$

Observe that

$$S = s\partial_s + q\partial_q,\ \Omega_{12} = \partial_\theta,\ \Omega_{01} = \cos(\theta)(s\partial_s - q\partial_q) - \frac{t}{r}\sin(\theta)\partial_\theta,\ \Omega_{02} = \sin(\theta)(s\partial_s - q\partial_q) + \frac{t}{r}\cos(\theta)\partial_\theta.$$

Hence Proposition 3.7.2 is an immediate consequence of Proposition (3.7.3).

Proposition 3.7.3. *We assume Let $j+k+l \leq N-5$ then in the region $r > \frac{t}{2}$ we have the estimates on h_0, for $q < 0$, if $j,k \geq 1$*

$$|\partial_s^j \partial_q^k \partial_\theta^l h_0| \lesssim \frac{\varepsilon^2}{s^{j+\frac{1}{2}}(1+|q|)^{k+1-4\rho}}$$

and

$$|\partial_q^k\partial_\theta^l h_0| \lesssim \frac{\varepsilon^2}{(1+|q|)^k}\left(\frac{1}{\sqrt{1+s}}+\frac{1}{(1+|q|)^{1-4\rho}}\right), \quad |\partial_s^j\partial_\theta^l h_0| \lesssim \frac{\varepsilon^2}{(1+s)^{\frac32+j}}.$$

For $q>0$ we have, with $j\geq 1$

$$|\partial_s^j\partial_q^k\partial_\theta^l h_0| \lesssim \frac{\varepsilon^2}{s^{j+\frac12}(1+|q|)^{k+\frac32+2(\delta-\sigma)}},\ |\partial_q^k\partial_\theta^l h_0| \lesssim \frac{\varepsilon^2}{(1+|q|)^{k+2+2(\delta-\sigma)}}.$$

Proof. We assume first $j=0$ and $k\geq 1$. We assume $l+k\leq N-3$. Then we can write

$$\partial_q^k\partial_\theta^l h_0 = -2\partial_q^{k-1}\partial_\theta^l\left(r(\partial_q\varphi)^2+\partial_q^2(q\chi(q))b(\theta)\right).$$

Therefore we can estimate

$$|\partial_q^k\partial_\theta^l h_0| \lesssim \frac{r}{(1+|q|)^{k-1}}\sum_{J\leq k+l-1}|Z^J(\partial_q\varphi)^2| + \frac{1}{(1+|q|)^{k+1}}|\partial_\theta^l b|.$$

The terms in $Z^J(\partial_q\varphi)^2$ are of the form $\partial_q Z^{J_1}\varphi\partial_q Z^{J_2}\varphi$, where $J_1\leq \frac{k+l}{2}\leq N-15$ therefore we can estimate, thanks to the bootstrap assumption (3.4.15)

$$|\partial_q Z^{J_1}\varphi| \lesssim \frac{\varepsilon}{(1+|q|)^{\frac32-4\rho}\sqrt{1+s}},$$

and we estimate $\partial_q Z^{J_2}\varphi$ thanks to (3.4.28) of Proposition 3.4.8 since $J_2\leq l+k-1\leq N-4$

$$|\partial_q Z^{J_2}\varphi| \lesssim \frac{\varepsilon}{\sqrt{1+|q|}\sqrt{1+s}}.$$

Consequently we have shown that for $k+l\leq N-3$, $k\geq 1$

$$|\partial_q^k\partial_\theta^l h_0| \lesssim \frac{\varepsilon^2}{(1+|q|)^{k+1-4\rho}}. \tag{3.7.12}$$

In the region $q>0$ we have the better estimate for $i=1,2$ thanks to (3.4.31) of Proposition 3.4.8

$$|\partial_q Z^{J_i}\varphi| \lesssim \frac{\varepsilon}{(1+|q|)^{\frac32+\delta-\sigma}\sqrt{1+s}},$$

so

$$|\partial_q^k\partial_\theta^l h_0| \lesssim \frac{\varepsilon^2}{(1+|q|)^{k+2+2\delta-2\sigma}}. \tag{3.7.13}$$

We now assume $k\geq 1$, $j\geq 1$ and estimate $\partial_s^j\partial_q^k\partial_\theta^l h_0$, for $j+k+l\leq N-4$. Thanks to (3.7.3), we can write

$$\partial_s^j\partial_q^k\partial_\theta^l h_0 = -\partial_s^{j-1}\partial_q^{k-1}\partial_\theta^l\left(r\partial_q\varphi\left(\Box\varphi-\frac1r\partial_s\varphi-\frac{1}{r^2}\partial_\theta^2\varphi\right)\right).$$

We estimate

$$\begin{aligned}\left|\partial_s^{j-1}\partial_q^{k-1}\partial_\theta^l\left(r\partial_q\varphi\left(\frac1r\partial_s\varphi+\frac{1}{r^2}\partial_\theta^2\varphi\right)\right)\right| &\lesssim \frac{1}{(1+|q|)^{k-1}(1+|s|)^{j-1}}\left|Z^{j+k+l-2}\left(r\partial_q\varphi\left(\frac1r\partial_s\varphi+\frac{1}{r^2}\partial_\theta^2\varphi\right)\right)\right|\\ &\lesssim \frac{1}{(1+|q|)^k(1+|s|)^j}\sum_{\substack{J_1+J_2\\ \leq j+k+l-2}}|Z^{J_1+2}\varphi Z^{J_2+1}\varphi|\end{aligned}$$

We can assume $J_1\leq\frac{j+k+l-2}{2}$. In the region $q<0$, (3.4.15) and (3.4.35) yield

$$|Z^{J_1+2}\varphi| \lesssim \frac{\varepsilon}{\sqrt{1+s}(1+|q|)^{\frac12-4\rho}}, \quad |Z^{J_2}\varphi| \lesssim \frac{\varepsilon\sqrt{1+|q|}}{\sqrt{1+s}}.$$

Consequently, for $q<0$

$$\left|\partial_s^{j-1}\partial_q^{k-1}\partial_\theta^l\left(r\partial_q\varphi\left(\frac{1}{r}\partial_s\varphi+\frac{1}{r^2}\partial_\theta^2\varphi\right)\right)\right|\lesssim\frac{\varepsilon^2}{(1+|q|)^{k-4\rho}(1+s)^{j+1}}. \qquad (3.7.14)$$

To estimate the contribution of $\Box\varphi$, we write as before $\Box\varphi=(\Box-\Box_g)\varphi$. Following Remark 3.2.2, it is sufficient to estimate

$$\begin{aligned}\left|\partial_s^{j-1}\partial_q^{k-1}\partial_\theta^l\left(rg_{LL}\partial_q\varphi\partial_q^2\varphi\right)\right| &\lesssim \frac{1}{(1+|q|)^{k-1}(1+|s|)^{j-1}}\left|Z^{k+j+l-2}\left(rg_{LL}\partial_q\varphi\partial_q^2\varphi\right)\right|\\ &\lesssim \frac{1}{(1+|q|)^{k}(1+|s|)^{j-2}}\sum_{\substack{J_1+J_2+J_3\\ \leq j+k+l-2}}|Z^{J_1}g_{LL}\partial_qZ^{J_2+1}\varphi\partial_qZ^{J_3}\varphi|.\end{aligned}$$

We have $J_1+J_2+J_3\leq j+k+l-2\leq N-5$. We separate in two cases

- $J_1\leq\frac{N}{2}-2$ and $J_2\leq\frac{N}{2}-2$: then we have thanks to (3.5.10), (3.4.15) and (3.4.28)

$$|Z^{J_1}g_{LL}|\lesssim\frac{\varepsilon(1+|q|)^{\frac{3}{2}}}{(1+s)^{\frac{3}{2}}},$$

$$|\partial_qZ^{J_2+1}\varphi|\lesssim\frac{\varepsilon}{\sqrt{1+s}(1+|q|)^{\frac{3}{2}-4\rho}},\quad |\partial_qZ^{J_3}\varphi|\lesssim\frac{\varepsilon}{\sqrt{1+|q|}\sqrt{1+s}}.$$

  The case $J_1\leq\frac{N}{2}-2$ and $J_3\leq\frac{N}{2}-2$ can be treated in the same way.

- $J_2\leq\frac{N}{2}-2$ and $J_3\leq\frac{N}{2}-2$ then, since $|J_1|\leq j+k+l-2\leq N-4$ we have thanks to (3.5.11) and (3.4.15)

$$|Z^{J_1}g_{LL}|\lesssim\frac{\varepsilon(1+|q|)^{2+\mu}}{(1+s)^{\frac{3}{2}}},\quad |\partial_qZ^J\varphi|\lesssim\frac{\varepsilon}{\sqrt{1+s}(1+|q|)^{\frac{3}{2}-4\rho}},\ for\ J=J_2+1,J_3.$$

In the first case we obtain

$$|Z^{J_1}g_{LL}\partial_qZ^{J_2+1}\varphi\partial_qZ^{J_3}\varphi|\lesssim\frac{\varepsilon^3}{(1+s)^{\frac{5}{2}}(1+|q|)^{\frac{1}{2}-4\rho}}, \qquad (3.7.15)$$

and in the last case we obtain

$$|Z^{J_1}g_{LL}\partial_qZ^{J_2+1}\varphi\partial_qZ^{J_3}\varphi|\lesssim\frac{\varepsilon^3}{(1+s)^{\frac{5}{2}}(1+|q|)^{1-8\rho-\mu}}.$$

We have $\mu+4\rho\leq\frac{1}{2}$. Consequently, we have in the region $q<0$

$$\left|\partial_s^{j-1}\partial_q^{k-1}\partial_\theta^l\left(rg_{LL}\partial_q\varphi\partial_q^2\varphi\right)\right|\lesssim\frac{\varepsilon^3}{(1+|q|)^{k+\frac{1}{2}-4\rho}(1+|s|)^{j-2}}. \qquad (3.7.16)$$

Estimates (3.7.14) and (3.7.16) yield, in the region $q<0$ for $j+k+l\leq N-4$, $j,k\geq1$

$$|\partial_s^j\partial_q^k\partial_\theta^lh_0|\lesssim\frac{\varepsilon^2}{(1+s)^{j+\frac{1}{2}}(1+|q|)^{k+\frac{1}{2}-4\rho}}. \qquad (3.7.17)$$

In the region $q>0$, thanks to (3.4.31) and (3.5.12) we have the better estimate, for $J\leq N-5$

$$|\partial_qZ^J\varphi|\leq\frac{\varepsilon}{\sqrt{1+s}(1+|q|)^{\frac{3}{2}+\delta-\sigma}},\ |Z^Jg_{LL}|\leq\frac{\varepsilon(1+|q|)^{\frac{1}{2}-\delta+\sigma}}{(1+s)^{\frac{3}{2}}},$$

so we have

$$|\partial_s^j \partial_q^k \partial_\theta^l h_0| \lesssim \frac{\varepsilon^2}{(1+s)^{j+\frac{1}{2}}(1+|q|)^{k+\frac{3}{2}+2(\delta-\sigma)}}. \tag{3.7.18}$$

We now assume $k=0$ and $j \geq 1$. We obtain an estimate on $\partial_s^j \partial_\theta^l h_0$ for $q>0$ by integrating (3.7.18) for $k=1$ with respect to q, from the hypersurface $t=0$. We obtain for $j+l \leq N-4$, $j \geq 1$, $q>0$

$$|\partial_s^j \partial_\theta^l h_0| \lesssim \frac{\varepsilon^2}{(1+s)^{j+\frac{1}{2}}(1+|q|)^{\frac{3}{2}+2(\delta-\sigma)}}. \tag{3.7.19}$$

For $q<0$, we integrate (3.7.17) from $q=0$. We obtain for $j+l \leq N-4$, $j \geq 1$,

$$|\partial_s^j \partial_\theta^l h_0| \lesssim \frac{\varepsilon^2}{(1+s)^{j+\frac{1}{2}}}. \tag{3.7.20}$$

We now estimate $\partial_\theta^l h_0$ for $l \leq N-5$. Recall from Corollary 3.6.4 that

$$\left|\partial_\theta^l \left(b(\theta) + \int_{\Sigma_{T,\theta}} (\partial_q \varphi)^2 r dr\right)\right| \lesssim \frac{\varepsilon^2}{T^{\frac{1}{2}}}.$$

Moreover, we can write, thanks to the estimate (3.7.20)

$$\partial_r \partial_\theta^l h_0 = \partial_q \partial_\theta^l h_0 + \partial_s \partial_\theta^l h_0 = \partial_\theta^l \left(-2r(\partial_q\varphi)^2\right) - 2\partial_q^2(q\chi(q))\partial_\theta^l b + O\left(\frac{\varepsilon^2}{(1+s)^{\frac{3}{2}}}\right).$$

Therefore, by integrating this on the line $t=T$, we have

$$\partial_\theta^l h_0(T,R,\theta) = \int_0^R \partial_q \partial_\theta^l h_0 dr + O\left(\frac{\varepsilon^2}{\sqrt{1+T}}\right),$$

and consequently, thanks to (3.7.12) we have the estimate, for $l \leq N-4$ and $q<0$

$$|\partial_\theta^l h_0(T,r,\theta)| \lesssim \frac{\varepsilon^2}{\sqrt{1+T}} + \frac{\varepsilon^2}{(1+|q|)^{1-4\rho}}.$$

To have an estimate everywhere, we integrate (3.7.20) for $j=0$ with respect to s, as shown in the figure 3.7.1. We obtain, for $l \leq N-5$

$$|\partial_\theta^l h_0| \lesssim \frac{\varepsilon^2}{\sqrt{1+s}} + \frac{\varepsilon^2}{(1+|q|)^{1-4\rho}}. \tag{3.7.21}$$

In the region $q>0$, we just integrate (3.7.19) from $t=0$, and we obtain

$$|\partial_\theta^l h_0| \lesssim \frac{\varepsilon^2}{(1+|q|)^{2+2(\delta-\sigma)}}. \tag{3.7.22}$$

In view of (3.7.17), (3.7.18), (3.7.12), (3.7.13), (3.7.20), (3.7.19), (3.7.21), (3.7.22) we conclude the proof of Proposition 3.7.3. □

3.7.2 Estimation of $\Box\Upsilon(\frac{r}{t})h_0$

Proposition 3.7.4. *Let $I \leq N-7$ We have the estimate for $q<0$*

$$\left|Z^I\left(\Box\left(\Upsilon\left(\frac{r}{t}\right)h_0\right) - \Upsilon\left(\frac{r}{t}\right)\left(-2(\partial_q\varphi)^2 - 2\frac{b(\theta)\partial_q^2(q\chi(q))}{r}\right)\right)\right| \lesssim \frac{\varepsilon^2}{(1+s)^{\frac{3}{2}}(1+|q|)},$$

and for $q>0$

$$\left|Z^I\left(\Box\left(\Upsilon\left(\frac{r}{t}\right)h_0\right) - \Upsilon\left(\frac{r}{t}\right)\left(-2(\partial_q\varphi)^2 - 2\frac{b(\theta)\partial_q^2(q\chi(q))}{r}\right)\right)\right| \lesssim \frac{\varepsilon^2}{(1+s)^{\frac{3}{2}}(1+|q|)^{2+2(\delta-\sigma)}}.$$

Proof. We have in view of (3.4.10), (3.7.2) and (3.7.3),

$$\begin{aligned}\Box\left(\Upsilon\left(\frac{r}{t}\right)h_0\right)=&\Upsilon\left(\frac{r}{t}\right)\left(4\partial_s\partial_q h_0+\frac{1}{r}(\partial_s h_0+\partial_q h_0)+\frac{1}{r^2}\partial_\theta^2 h_0\right)+\nabla\Upsilon\left(\frac{r}{t}\right).\nabla h_0+h_0\Box\Upsilon\left(\frac{r}{t}\right)\\ =&\Upsilon\left(\frac{r}{t}\right)\left(-4r\partial_q\varphi\left(\Box\varphi-\frac{1}{r}\partial_s\varphi-\frac{1}{r^2}\partial_\theta^2\varphi\right)+\frac{1}{r}\left(-2(\partial_q\varphi)^2r-2b(\theta)\partial_q^2(q\chi(q))\right)\right)\\ &+\Upsilon\left(\frac{r}{t}\right)\left(\frac{1}{r}\partial_s h_0+\frac{1}{r^2}\partial_\theta^2 h_0\right)+\nabla\Upsilon\left(\frac{r}{t}\right).\nabla h_0+h_0\Box\Upsilon\left(\frac{r}{t}\right)\\ =&\Upsilon\left(\frac{r}{t}\right)\left(-2(\partial_q\varphi)^2-4\frac{b(\theta)\partial_q^2(q\chi(q))}{r}\right)-4r\Upsilon\left(\frac{r}{t}\right)\partial_q\varphi\Box\varphi+f(s,q,\theta),\end{aligned}$$

where

$$f(s,q,\theta)=\Upsilon\left(\frac{r}{t}\right)\left(4\partial_q\varphi\left(\partial_s\varphi+\frac{\partial_\theta^2\varphi}{r}\right)+\frac{1}{r}\partial_s h_0+\frac{1}{r^2}\partial_\theta^2 h_0\right)+\nabla\Upsilon\left(\frac{r}{t}\right).\nabla h_0+h_0\Box\Upsilon\left(\frac{r}{t}\right).$$

We can estimate $Z^I f$, noticing that when $\Upsilon'\left(\frac{r}{t}\right)\neq 0$ we have $r\sim t\sim|q|$. We obtain

$$|Z^I f|\lesssim\frac{1}{(1+s)^2}\sum_{J\leq I+2}|Z^I h_0|+\frac{1}{1+s}\sum_{I_1+I_2\leq I}|Z^{I_1+2}\varphi||\partial_q Z^{I_2}\varphi|.$$

Proposition 3.7.2 yields, for $I\leq N-7$

$$\frac{1}{(1+s)^2}\sum_{J\leq I+2}|Z^I h_0|\lesssim\frac{\varepsilon^2}{(1+s)^2\sqrt{1+|q|}},$$

and as usual we may estimate, thanks to (3.4.28) and (3.4.15),

$$|Z^{I_1+2}\varphi\partial_q Z^{I_2}\varphi|\lesssim\frac{\varepsilon^2}{(1+s)(1+|q|)^{1-4\rho}},$$

therefore we obtain

$$|Z^I f|\lesssim\frac{\varepsilon^2}{(1+s)^2\sqrt{1+|q|}}.$$

In the region $q>0$, we have the better estimate

$$|Z^I f|\lesssim\frac{\varepsilon^2}{(1+s)^2(1+|q|)^{2+2(\delta-\sigma)}}.$$

To estimate $\Box\varphi$ we write, as before

$$\Box\varphi=\Box\varphi-\Box_g\varphi.$$

It is sufficient to estimate a term of the form $g_{LL}\partial_q^2\varphi$. Therefore we write, like in estimate (3.7.15),

$$|Z^I(rg_{LL}\partial_q\varphi\partial_q^2\varphi)|\lesssim\frac{r}{1+|q|}\sum_{J_1+J_2+J_3\leq I}|Z^{J_1}g_{LL}||\partial Z^{J_2+1}\varphi||\partial Z^{J_3}\varphi|\lesssim\frac{\varepsilon^3}{(1+s)^{\frac{3}{2}}(1+|q|)^{\frac{3}{2}-4\rho}}.$$

In the region $q>0$, we have the better estimate

$$|Z^I(r\partial_q\varphi\Box\varphi)|\lesssim\frac{\varepsilon^3}{(1+s)^{\frac{3}{2}}(1+|q|)^{2+2(\delta-\sigma)}}.$$

This concludes the proof of Proposition 3.7.4. $\Box$

3.7.3 Estimation on $\widetilde{h}$

We recall that $\widetilde{h}$ satisfies the equation

$$\begin{cases} \Box\widetilde{h} = \Box\left(\Upsilon\left(\frac{r}{t}\right)h_0\right) + \Upsilon\left(\frac{r}{t}\right)g_{LL}\partial_q^2 h_0 + 2\Upsilon\left(\frac{r}{t}\right)(\partial_q\varphi)^2 - 2(R_b)_{qq} + \Upsilon\left(\frac{r}{t}\right)\widetilde{Q}_{\underline{L}\underline{L}}(h_0,\widetilde{g}), \\ (\widetilde{h},\partial_t\widetilde{h})|_{t=0} = (0,0), \end{cases}$$

where $\widetilde{Q}_{\underline{L}\underline{L}}$ is defined by (3.4.12).

Proposition 3.7.5. $\widetilde{h}$ *satisfies, for* $I \le N-7$

$$|Z^I\widetilde{h}| \lesssim \frac{\varepsilon^2}{(1+s)^{\frac{1}{2}-\rho}}.$$

Proof. Proposition 3.7.4 gives for $I \le N-7$ and $q<0$

$$\left|Z^I\left(\Box\Upsilon\left(\frac{r}{t}\right)h_0 + 2\Upsilon\left(\frac{r}{t}\right)(\partial_q\varphi)^2 - 2(R_b)_{qq}\right)\right| \lesssim \frac{\varepsilon^2}{(1+s)^{\frac{3}{2}}(1+|q|)}, \tag{3.7.23}$$

where we have used that thanks to (3.1.7)

$$\left|(R_b)_{qq} - 2\frac{b(\theta)\partial_q^2(q\chi(q))}{r}\right| \lesssim \frac{\mathbb{1}_{1\le q\le 2}\varepsilon^2}{(1+s)^2}.$$

To estimate $Z^I(g_{LL}\partial_q^2 h_0)$ we use the transport equation for h_0

$$g_{LL}\partial_q^2 h_0 = g_{LL}\partial_q(-2r(\partial_q\varphi)^2 - 2b(\theta)\partial_q^2(q\chi(q))$$

We estimate the first term as in the proof of Proposition 3.7.4.

$$|Z^I(rg_{LL}\partial_q\varphi\partial_q^2\varphi)| \lesssim \frac{\varepsilon^2}{(1+s)^{\frac{3}{2}}(1+|q|)^{\frac{3}{2}-4\rho}}.$$

To estimate the second term, we note that the terms of the form $\chi^{(j)}(q)$ decay faster than any power of q, so thanks to (3.5.11),

$$|Z^I\left(g_{LL}b(\theta)\partial_q^2(q\chi(q))\right)| \lesssim \frac{\varepsilon^2}{s^{\frac{3}{2}}(1+|q|)^3}.$$

Consequently we have proved

$$\left|Z^i\left(\Upsilon\left(\frac{r}{t}\right)g_{LL}\partial_q^2 h_0\right)\right| \lesssim \frac{\varepsilon^2}{(1+s)^{\frac{3}{2}}(1+|q|)^{\frac{3}{2}-4\rho}}. \tag{3.7.24}$$

We now estimate $\widetilde{Q}_{\underline{L}\underline{L}}(h_0,\widetilde{g})$. We note than in the region $q<0$ the only term is $\partial_{\underline{L}}\widetilde{g}_{LL}\partial_{\underline{L}}h_0$. We use again the transport equation for h_0

$$\partial_q g_{LL}\partial_q h_0 = \partial_q g_{LL}(-2r(\partial_q\varphi)^2 - 2b(\theta)\partial_q^2(q\chi(q)).$$

Consequently, for similar reasons than for (3.7.24), we obtain in the region $q<0$

$$\left|Z^i\left(\Upsilon\left(\frac{r}{t}\right)\partial_q g_{LL}\partial_q h_0\right)\right| \lesssim \frac{\varepsilon^2}{(1+s)^{\frac{3}{2}}(1+|q|)^{\frac{3}{2}-4\rho}}. \tag{3.7.25}$$

Thanks to (3.7.23), (3.7.24) and (3.7.25), we have in the region $q < 0$ for $I \leq N-7$

$$|\Box Z^I \widetilde{h}| \lesssim \frac{\varepsilon^2}{(1+s)^{\frac{3}{2}}(1+|q|)}. \tag{3.7.26}$$

In the region $q > 0$, we have to estimate in $\widetilde{Q}_{\underline{L}\underline{L}}(h_0, \widetilde{g})$ the term $\partial_{\underline{L}}(g_b)_{UU}\partial_{\underline{L}} g_{\underline{L}L}$, which is of the form $\frac{\chi(q)b(\theta)}{r}\partial_q g_{\underline{L}L}$. Thanks to (3.4.32) we have

$$\left| Z^I \left(\frac{\chi(q)b(\theta)}{r}\partial_q g_{\underline{L}L} \right) \right| \lesssim \frac{\varepsilon^2}{(1+s)^{\frac{3}{2}}(1+|q|)^{\frac{3}{2}+\delta-\sigma}}. \tag{3.7.27}$$

The other terms give contributions similar to the one of Proposition 3.7.4. Consequently, for $q > 0$ we have the better estimate for $I \leq N-7$

$$|\Box Z^I \widetilde{h}| \lesssim \frac{\varepsilon^2}{(1+s)^{\frac{3}{2}}(1+|q|)^{\frac{3}{2}+\delta-\sigma}}. \tag{3.7.28}$$

We now use lemma 3.7.6, whose proof is given at the end of this section, to conclude.

Lemma 3.7.6. *Let $\beta, \alpha \geq 0$, such that $\beta - \alpha \geq \rho > 0$. Let u be such that*

$$|\Box u| \lesssim \frac{1}{(1+s)^{\frac{3}{2}-\alpha}(1+|q|)}, \; for\ q<0 \quad |\Box u| \lesssim \frac{1}{(1+s)^{\frac{3}{2}-\alpha}(1+|q|)^{1+\beta}}, \; for\ q>0,$$

and $(u, \partial_t u)|_{t=0} = 0$. Then we have the estimate

$$|u| \lesssim \frac{(1+t)^{\alpha+\rho}}{\sqrt{1+s}}.$$

Thanks to (3.7.26) and (3.7.28), the conditions of Lemma 3.7.6 are satisfied with $\alpha = 0$ and $\beta = \frac{1}{2} + \delta - \sigma$. Moreover, the initial data for $Z^I \widetilde{h}$ are given by the right-hand side of (3.4.11) (i.e. they are quadratic), therefore, for $I \leq N-7$ at $t=0$ we have

$$|Z^I \widetilde{h}| + (1+r)|\partial_t Z^I \widetilde{h}| \lesssim \frac{\varepsilon^2}{(1+r)^{1+\delta}}.$$

Consequently, Lemma 3.7.6 and Proposition 3.1.5 yield for $I \leq N-7$

$$|Z^I \widetilde{h}| \lesssim \frac{\varepsilon^2 (1+t)^{\rho}}{\sqrt{1+s}}.$$

This concludes the proof of Lemma 3.7.5. □

Proof of Lemma 3.7.6. Let $t_0 > 0$. We consider times $t \leq t_0$. In the region $r \leq 2t$ we have $|q| \leq t \leq t_0$ and $s \leq 3t \leq 3t_0$. Therefore

$$|\Box u| \lesssim \frac{(1+t_0)^{\alpha+\rho}}{(1+|q|)^{1+\frac{\rho}{2}}(1+s)^{\frac{3}{2}+\frac{\rho}{2}}}.$$

In the region $r \leq 2t$, we have $\frac{r}{2} \leq |q| \leq r$ and $r \leq s \leq \frac{3r}{2}$, therefore

$$|\Box u| \lesssim \frac{1}{(1+r)^{\frac{3}{2}-\alpha+1+\beta}} \lesssim \frac{(1+t_0)^{\alpha+\rho}}{(1+r)^{\frac{5}{2}-\alpha+\beta}} \lesssim \frac{(1+t_0)^{\alpha+\rho}}{(1+|q|)^{1+\frac{\rho}{2}}(1+s)^{\frac{3}{2}+\frac{\rho}{2}}},$$

provided $\frac{5}{2} + \rho \leq \frac{5}{2} + \beta - \alpha$, i.e. $\beta - \alpha \geq \rho$. Consequently, the $L^\infty - L^\infty$ estimate yields, for $t \leq t_0$

$$|u| \lesssim \frac{(1+t_0)^{\alpha+\rho}}{\sqrt{1+s}}.$$

If we take $t = t_0$ we have proved

$$|u| \lesssim \frac{(1+t)^{\alpha+\rho}}{\sqrt{1+s}},$$

which concludes the proof of Lemma 3.7.6. □

3.8 Commutation with the vector fields and L^∞ estimates

3.8.1 Estimates for $I \leq N-14$

Proposition 3.8.1. *We have the estimates for for $I \leq N-14$*

$$|Z^I \widetilde{g}_1| \leq \frac{C_0\varepsilon + C\varepsilon^2}{(1+s)^{\frac{1}{2}-\rho}},$$
$$|Z^I \varphi| \leq \frac{C_0\varepsilon + C\varepsilon^2}{\sqrt{1+s}(1+|q|)^{\frac{1}{2}-4\rho}}.$$

This proposition is a consequence of $L^\infty - L^\infty$ estimates and the following propositions.

Proposition 3.8.2. *We have the estimate for $I \leq N-14$*

$$|\Box Z^I \varphi| \lesssim \frac{\varepsilon^2}{(1+s)^{2-3\rho}(1+|q|)}, \; q<0,$$
$$|\Box Z^I \varphi| \lesssim \frac{\varepsilon^2}{(1+s)^{2}(1+|q|)^{1+\delta-\sigma}}, \; q>0.$$

Proposition 3.8.3. *We have the estimate for $I \leq N-14$*

$$|\Box Z^I \widetilde{g}_1| \lesssim \frac{\varepsilon^2}{(1+s)^{\frac{3}{2}}(1+|q|)}, \; q<0,$$
$$|\Box Z^I \widetilde{g}_1| \lesssim \frac{\varepsilon^2}{(1+s)^{\frac{3}{2}}(1+|q|)^{\frac{3}{2}+\delta-\sigma}}, \; q>0.$$

We first assume Proposition 3.8.2 and 3.8.3, and prove Proposition 3.8.1.

Proof of Proposition 3.8.1. We have

$$|\Box Z^I \varphi| \lesssim \frac{\varepsilon^2}{(1+s)^{2-3\rho}(1+|q|)} \lesssim \frac{\varepsilon^2}{(1+s)^{2-4\rho}(1+|q|)^{1+\rho}},$$

therefore the $L^\infty - L^\infty$ estimate, combined with Proposition 3.1.5 for the contribution of the initial data yields

$$|Z^I \varphi| \leq \frac{C_0\varepsilon}{\sqrt{1+s}\sqrt{1+|q|}} + \frac{C\varepsilon^2}{\sqrt{1+s}(1+|q|)^{\frac{1}{2}-4\rho}},$$

where C is a constant depending on ρ.

The estimate $\widetilde{g}_1$ follows from Lemma 3.7.6 with $\alpha=0$, $\beta=\frac{3}{2}+\delta-\sigma$ combined with Proposition 3.1.5

$$|Z^I \widetilde{g}_1| \leq \frac{C_0\varepsilon}{\sqrt{1+s}\sqrt{1+|q|}} + \frac{C\varepsilon^2}{(1+s)^{\frac{1}{2}-\rho}},$$

which concludes the proof of Proposition 3.8.1. □

Proof of Proposition 3.8.2. We first estimate $\Box Z^I \varphi$ in the region $q<0$

$$Z^I \Box \varphi = Z^I\left(\Box\varphi - \Box_g \varphi\right).$$

In the region $q<0$, thanks to Remark 3.2.2, it is sufficient to estimate $Z^I\left(g_{LL}\partial_q^2\varphi\right)$

$$|Z^{I-J} g_{LL}\partial_q^2 Z^J \varphi| \lesssim \frac{1}{(1+|q|)^2}|Z^{I-J}g_{LL}||Z^{J+2}\varphi|.$$

If $J \leq \frac{N-14}{2}$ we have $J+2 \leq \frac{N-14}{2}+2 \leq N-14$ so, thanks to (3.4.15)

$$|Z^{J+2}\varphi| \lesssim \frac{\varepsilon}{(1+s)^{\frac{1}{2}}(1+|q|)^{\frac{1}{2}-4\rho}},$$

and since $I-J \leq N-14$ we have thanks to (3.5.9)

$$|Z^{I-J}g_{LL}| \lesssim \frac{\varepsilon(1+|q|)}{(1+s)^{\frac{3}{2}-2\rho}}.$$

Therefore

$$|Z^{I-J}g_{LL}\partial_q^2 Z^J\varphi| \lesssim \frac{\varepsilon^2}{(1+s)^{2-2\rho}(1+|q|)^{\frac{3}{2}-4\rho}}.$$

If $I-J \leq \frac{N-14}{2} \leq N-15$ we have thanks to (3.5.8)

$$|Z^{I-J}g_{LL}| \lesssim \frac{\varepsilon(1+|q|)}{(1+s)^{\frac{3}{2}-\rho}},$$

and since $J+2 \leq N-12$ we have thanks to (3.4.17)

$$|Z^{J+2}\varphi| \lesssim \frac{\varepsilon}{(1+s)^{\frac{1}{2}-2\rho}}.$$

In the two cases, we have for $q<0$

$$|Z^{I-J}g_{LL}\partial_q^2 Z^J\varphi| \lesssim \frac{\varepsilon^2}{(1+s)^{2-3\rho}(1+|q|)}. \tag{3.8.1}$$

In the region $q>0$ we have the better estimate thanks to (3.5.12) and (3.4.32)

$$|Z^{I-J}g_{LL}\partial_q^2 Z^J\varphi| \lesssim \frac{\varepsilon^2}{(1+s)^{2}(1+|q|)^{2+2(\delta-\sigma)}}. \tag{3.8.2}$$

In the region $q>0$ we also have to take into account the crossed term. These terms are described by (3.2.13) in Section 3.2.5. It is sufficient to estimate

$$Z^I\left(b(\theta)\frac{\chi(q)}{r}\partial_s\varphi\right).$$

Since they occur only in the region $q>0$, we can estimate, thanks to (3.4.38)

$$|Z^I\varphi| \lesssim \frac{\varepsilon}{\sqrt{1+s}(1+|q|)^{\frac{1}{2}+\delta-\sigma}}.$$

Therefore

$$\left|Z^I b(\theta)\frac{\partial_q(q\chi(q))}{r}\partial_s\varphi\right| \lesssim \frac{\varepsilon^2}{(1+s)^{\frac{5}{2}}(1+|q|)^{\frac{1}{2}+\delta-\sigma}} \lesssim \frac{\varepsilon^2}{(1+s)^{2}(1+|q|)^{1+\delta-\sigma}}. \tag{3.8.3}$$

Estimates (3.8.1), (3.8.2) and (3.8.3) conclude the proof of Proposition 3.8.2. □

Proof of Proposition 3.8.3. We write the equation for $\widetilde{g}_1$. We have, thanks to (3.2.4) and (3.2.7)

$$\begin{aligned}\Box(\widetilde{g}_1)_{\mu\nu} =& -2\partial_\mu\varphi\partial_\nu\varphi + 2(R_b)_{\mu\nu} + (dq^2)_{\mu\nu}\Box\Upsilon\left(\frac{r}{t}\right)h_0 \\ &+ \Upsilon\left(\frac{r}{t}\right)\frac{1}{r^2}\left(u^1_{\mu\nu}(\theta)h_0 + u^2_{\mu\nu}(\theta)\partial_\theta h_0\right) \\ &+ P_{\mu\nu}(g)(\partial\widetilde{g},\partial\widetilde{g}) + \widetilde{P}_{\mu\nu}(\widetilde{g},g_b),\end{aligned} \tag{3.8.4}$$

and therefore $\Box Z^I(\widetilde{g}_1)_{\mu\nu} = f_{\mu\nu}$, where the terms in $f_{\mu\nu}$ are of the forms

- the quasilinear terms : thanks to Remark 3.2.2 it is sufficient to study $Z^I(g_{LL}\partial_q^2\widetilde{g}_1)$,
- the terms coming from the non commutation of the wave operator with the null decomposition: they are calculated in (3.2.7) and they are of the form $\Upsilon(\frac{r}{t})\frac{1}{r^2}\partial_\theta Z^I h_0$,
- the semi-linear terms: following section the worst term is the term $Z^I\left(\partial_{\underline{L}} g_{\underline{LL}} \partial_{\underline{L}} g_{LL}\right)$ appearing in $Z^I P_{\underline{LL}}$ (see (3.2.9)).
- the crossed terms with the background metric g_b: the worst term is the term $Z^I\left(\partial_{\underline{L}}(g_b)_{UU}\partial_{\underline{L}} g_{L\underline{L}}\right)$ appearing in $Z^I \widetilde{P}_{\underline{LL}}$ (see (3.2.11)).

The quasilinear terms We estimate

$$Z^I\left(g_{LL}\partial_q^2\widetilde{g}_1\right)=\sum_{J\leq I} Z^{I-J}g_{LL}Z^J\partial_q^2\widetilde{g}_1.$$

We have

$$|Z^{I-J}g_{LL}\partial_q^2 Z^J\widetilde{g}_1|\lesssim \frac{1}{(1+|q|)^2}||Z^{I-J}g_{LL}||Z^{J+2}\widetilde{g}_1|.$$

If $J\leq \frac{N-14}{2}$ we have $J+2\leq \frac{N-14}{2}+2\leq N-14$ so thanks to (3.4.16)

$$|Z^{J+2}\widetilde{g}_1|\lesssim \frac{\varepsilon}{(1+s)^{\frac{1}{2}-\rho}},$$

and since $I-J\leq N-14$ we have thanks to (3.5.9)

$$|Z^{I-J}g_{LL}|\lesssim \frac{\varepsilon(1+|q|)}{(1+s)^{\frac{3}{2}-2\rho}}.$$

If $I-J\leq \frac{N-14}{2}\leq N-15$ we have thanks to (3.5.8)

$$|Z^{I-J}g_{LL}|\lesssim \frac{\varepsilon(1+|q|)}{(1+s)^{\frac{3}{2}-\rho}},$$

and since $J+2\leq N-12$ we have thanks to (3.4.18)

$$|Z^{J+2}\widetilde{g}_1|\lesssim \frac{\varepsilon}{(1+s)^{\frac{1}{2}-2\rho}}$$

In the two cases, we have

$$|Z^{I-J}g_{LL}\partial_q^2 Z^J\widetilde{g}_1|\lesssim \frac{\varepsilon^2}{(1+s)^{2-3\rho}(1+|q|)}. \tag{3.8.5}$$

The term coming from the non commutation of the wave operator with the null structure We have to estimate

$$\Upsilon\left(\frac{r}{t}\right)\frac{\partial_\theta Z^I h_0}{r^2}.$$

Since $I\leq N-14$, we have $I+1\leq N-5$ so thanks to Proposition 3.7.2

$$\left|\Upsilon\left(\frac{r}{t}\right)\frac{\partial_\theta Z^I h_0}{r^2}\right|\lesssim \frac{\varepsilon^2}{(1+s)^2\sqrt{1+|q|}}\lesssim \frac{\varepsilon^2}{(1+s)^{\frac{3}{2}}(1+|q|)}. \tag{3.8.6}$$

The semi-linear terms We estimate $Z^I\left(\partial_{\underline{L}} g_{\underline{L}\underline{L}} \partial_{\underline{L}} g_{LL}\right)$. For this, we have to estimate, using the decomposition (3.4.6)

$$Z^I\left(\partial_{\underline{L}} h_0 \partial_{\underline{L}} g_{LL}\right) \quad and \quad Z^I\left(\partial_{\underline{L}} \widetilde{g}_1 \partial_{\underline{L}} g_{LL}\right)$$

The first term has been estimated in (3.7.23). For the second term, we write

$$|Z^I\left(\partial_{\underline{L}} \widetilde{g}_1 \partial_{\underline{L}} g_{LL}\right)| \lesssim \frac{1}{1+|q|}\sum_{J\leq I}|Z^{J+1}\widetilde{g}_1||\partial Z^{I-J} g_{LL}|,$$

and we estimate if $J \leq \frac{N-14}{2}$ thanks to (3.4.16) and (3.5.4)

$$|Z^{J+1}\widetilde{g}_1| \lesssim \frac{\varepsilon}{(1+s)^{\frac{1}{2}-\rho}} \quad and \quad |\partial Z^{I-J} g_{LL}| \lesssim \frac{\varepsilon}{(1+s)^{\frac{3}{2}-2\rho}}.$$

If $I - J \leq \frac{N-14}{2}$ thanks to (3.4.18) and (3.5.3) we have

$$|Z^{J+1}\widetilde{g}_1| \lesssim \frac{\varepsilon}{(1+s)^{\frac{1}{2}-2\rho}} \quad and \quad |\partial Z^{I-J} g_{LL}| \lesssim \frac{\varepsilon}{(1+s)^{\frac{3}{2}-\rho}}.$$

In the two cases we have

$$|Z^I\left(\partial_{\underline{L}} \widetilde{g}_1 \partial_{\underline{L}} g_{LL}\right)| \lesssim \frac{\varepsilon^2}{(1+s)^{2-3\rho}(1+|q|)}.$$

This estimate and (3.7.23) yields for $I \leq N-14$

$$|Z^I\left(\partial_{\underline{L}} \widetilde{g}_1 \partial_{\underline{L}} g_{LL}\right)| \lesssim \frac{\varepsilon^2}{(1+s)^{\frac{3}{2}}(1+|q|)^{\frac{3}{2}-4\rho}}. \tag{3.8.7}$$

We have now estimated $\Box Z^I(\widetilde{g}_1)_{\mu\nu}$ in the region $q<0$. Thanks to (3.8.5), (3.8.6) and (3.8.7) we have, for $q<0$ and $I \leq N-14$

$$|\Box Z^I \widetilde{g}_1| \lesssim \frac{\varepsilon^2}{(1+s)^{2-3\rho}(1+|q|)} \tag{3.8.8}$$

The crossed terms The crossed term are only present in the region $q>0$. The estimate of

$$Z^I\left(\partial_q (g_b)_{UU} \partial_q \widetilde{g}_{\underline{L}L}\right)$$

is done in (3.7.27). The other terms give better contributions in the region $q>0$ (see Remark 3.2.3). Therefore we have for $q<0$ and $I \leq N-4$

$$|\Box Z^I \widetilde{g}_1| \lesssim \frac{\varepsilon^2}{(1+s)^{\frac{3}{2}}(1+|q|)^{\frac{3}{2}+\delta-\sigma}}. \tag{3.8.9}$$

The estimates (3.8.8) and (3.8.9) conclude the proof of Proposition 3.8.3. □

3.8.2 Estimates for $I \leq N-12$

Proposition 3.8.4. *We have the estimates for $I \leq N-12$*

$$|Z^I \varphi| \leq \frac{C_0\varepsilon + C\varepsilon^2}{(1+s)^{\frac{1}{2}-2\rho}},$$
$$|Z^I \widetilde{g}_1| \lesssim \frac{C_0\varepsilon + C\varepsilon^2}{(1+s)^{\frac{1}{2}-2\rho}}.$$

This proposition is a straightforward consequence of Lemma 3.7.6, Proposition 3.1.5 and the following propositions.

Proposition 3.8.5. *We have the estimate for $I \leq N-12$*

$$|\Box Z^I\varphi| \lesssim \frac{\varepsilon^2}{(1+s)^{\frac{3}{2}-\rho}(1+|q|)},\ q<0,$$
$$|\Box Z^I\varphi| \lesssim \frac{\varepsilon^2}{(1+s)^2(1+|q|)^{1+\delta-\sigma}},\ q>0.$$

Proposition 3.8.6. *We have the estimate for $I \leq N-12$*

$$|\Box Z^I\widetilde{g}_1| \lesssim \frac{\varepsilon^2}{(1+s)^{\frac{3}{2}-\rho}(1+|q|)},\ q<0,$$
$$|\Box Z^I\widetilde{g}_1| \lesssim \frac{\varepsilon^2}{(1+s)^{\frac{3}{2}}(1+|q|)^{\frac{3}{2}+\delta-\sigma}},\ q>0.$$

Proof of Proposition 3.8.5. We first estimate φ

$$Z^I\Box\varphi = Z^I\left(\Box\varphi - \Box_g\varphi\right).$$

In the region $q<0$, it is sufficient to estimate $Z^I\left(g_{LL}\partial_q^2\varphi\right)$

$$|Z^{I-J}g_{LL}\partial_q^2 Z^J\varphi| \lesssim \frac{1}{1+|q|}|Z^{I-J}g_{LL}||\partial_q Z^{J+1}\varphi|$$

If $J \leq \frac{N-12}{2}$ we have $J+1 \leq N-14$ so thanks to (3.4.15)

$$|\partial Z^{J+1}\varphi| \lesssim \frac{\varepsilon}{(1+s)^{\frac{1}{2}}(1+|q|)^{\frac{3}{2}-4\rho}},$$

and since $I-J \leq N-12$ we have thanks to (3.5.10)

$$|Z^{I-J}g_{LL}| \lesssim \frac{\varepsilon(1+|q|)}{1+s}.$$

Therefore

$$|Z^{I-J}g_{LL}\partial_q^2 Z^J\varphi| \lesssim \frac{\varepsilon^2}{(1+s)^{\frac{3}{2}}(1+|q|)^{\frac{3}{2}-4\rho}}.$$

If $I-J \leq \frac{N-12}{2} \leq N-15$ we have thanks to (3.5.8)

$$|Z^{I-J}g_{LL}| \lesssim \frac{\varepsilon(1+|q|)}{(1+s)^{\frac{3}{2}-\rho}},$$

and since $J+1 \leq N-12 \leq N-4$ we have thanks to (3.4.28)

$$|\partial Z^{J+1}\varphi| \lesssim \frac{\varepsilon}{\sqrt{1+s}\sqrt{(1+|q|)}}.$$

In the two cases, we have

$$|Z^{I-J}g_{LL}\partial_q^2 Z^J\widetilde{\varphi}| \lesssim \frac{\varepsilon^2}{(1+s)^{\frac{3}{2}-\rho}(1+|q|)}.$$

The main contribution in the region $q>0$ is like (3.8.3) in the proof of Proposition 3.8.2. This concludes the proof of Proposition 3.8.5. $\square$

Proof of Proposition 3.8.6. We estimate $\widetilde{g}_1$. We only deal with the quasilinear and semilinear terms in the region $q<0$, as the control obtain in the proof of Proposition 3.8.3 is sufficient to deal with the others (see (3.8.6) and (3.7.27)).

The semi-linear terms We estimate $Z^I\left(\partial_{\underline{L}} g_{\underline{LL}} \partial_{\underline{L}} g_{LL}\right)$. For this, we have to estimate

$$Z^I\left(\partial_{\underline{L}} h_0 \partial_{\underline{L}} g_{LL}\right) \quad and \quad Z^I\left(\partial_{\underline{L}} \widetilde{g}_1 \partial_{\underline{L}} g_{LL}\right)$$

The first term has been estimated in (3.7.23). For the second term, we write

$$|Z^I\left(\partial_{\underline{L}} \widetilde{g}_1 \partial_{\underline{L}} g_{LL}\right)| \lesssim \frac{1}{1+|q|} \sum_{J \leq I} |Z^{J+1} \widetilde{g}_1| |\partial Z^{I-J} g_{LL}|,$$

and we estimate if $J \leq \frac{N-12}{2}$ thanks to (3.4.16) and (3.5.5)

$$|Z^{J+1} \widetilde{g}_1| \lesssim \frac{\varepsilon}{(1+s)^{\frac{1}{2}-\rho}} \quad and \quad |\partial Z^{I-J} g_{LL}| \lesssim \frac{\varepsilon\sqrt{1+|q|}}{(1+s)^{\frac{3}{2}}}.$$

If $I-J \leq \frac{N-12}{2}$ thanks to (3.4.44) and (3.5.3) we have

$$|Z^{J+1} \widetilde{g}_1| \lesssim \varepsilon \quad and \quad |\partial Z^{I-J} g_{LL}| \lesssim \frac{\varepsilon}{(1+s)^{\frac{3}{2}-\rho}}.$$

In the two cases we have

$$|Z^I\left(\partial_{\underline{L}} \widetilde{g}_1 \partial_{\underline{L}} g_{LL}\right)| \lesssim \frac{\varepsilon^2}{(1+s)^{\frac{3}{2}-\rho}(1+|q|)}.$$

This estimate and (3.7.23) yields for $I \leq N-12$

$$|Z^I\left(\partial_{\underline{L}} \widetilde{g}_1 \partial_{\underline{L}} g_{LL}\right)| \lesssim \frac{\varepsilon^2}{(1+s)^{\frac{3}{2}-\rho}(1+|q|)}. \tag{3.8.10}$$

The quasilinear terms We estimate $Z^I\left(g_{LL} \partial_q^2 \widetilde{g}_1\right)$. We have

$$|Z^{I-J} g_{LL} \partial_q^2 Z^J \widetilde{g}_1| \lesssim \frac{1}{1+|q|} |Z^{I-J} g_{LL}| |\partial_q Z^{J+1} \widetilde{g}_1|.$$

If $J \leq \frac{N-12}{2}$ we have $J+2 \leq \frac{N-12}{2}+2 \leq N-14$ so thanks to (3.4.16)

$$|\partial_q Z^{J+1} \widetilde{g}_1| \lesssim \frac{\varepsilon}{(1+s)^{\frac{1}{2}-\rho}(1+|q|)},$$

and since $|I-J| \leq N-12$ we have thanks to (3.5.10)

$$|Z^{I-J} g_{LL}| \lesssim \frac{\varepsilon(1+|q|)}{1+s}.$$

If $|I-J| \leq \frac{N-12}{2} \leq N-15$ we have thanks to (3.5.8)

$$|Z^{I-J} g_{LL}| \lesssim \frac{\varepsilon(1+|q|)}{(1+s)^{\frac{3}{2}-\rho}}$$

and since $J+1 \leq N-11$ we have thanks to (3.4.44)

$$|\partial_q Z^{J+1} \widetilde{g}_1| \lesssim \frac{\varepsilon}{1+|q|}.$$

In the two cases, we have

$$|Z^{I-J} g_{LL} \partial_q^2 Z^J \widetilde{g}_1| \lesssim \frac{\varepsilon^2}{(1+s)^{\frac{3}{2}-\rho}(1+|q|)}. \tag{3.8.11}$$

The equation (3.8.10) and (3.8.11), together with (3.8.6) proved during the proof of Proposition 3.8.3 conclude the proof of Proposition 3.8.6 for $q<0$. The estimate for $\Box Z^I \widetilde{g}_1$ in the region $q>0$ is given by (3.8.9). This conclude the proof of Proposition 3.8.6 for $q>0$. $\Box$

3.9 Weighted energy estimate

We consider the equation

$$\Box_g u = f,$$

where $g = g_b + \widetilde{g}$ is our space-time metric, satisfying the bootstrap assumptions. We introduce the energy-momentum tensor associated to $\Box_g$

$$Q_{\alpha\beta} = \partial_\alpha u \partial_\beta u - \frac{1}{2} g_{\alpha\beta} g^{\mu\nu} \partial_\mu u \partial_\nu u.$$

We have

$$D^\alpha Q_{\alpha\beta} = f \partial_\beta u.$$

We also note $T = \partial_t$, and introduce the deformation tensor of T

$$\pi_{\alpha\beta} = D_\alpha T_\beta + D_\beta T_\alpha$$

where D is the covariant derivative. We have

$$D^\alpha (Q_{\alpha\beta} T^\beta) = f \partial_t u + Q_{\alpha\beta} \pi^{\alpha\beta}. \tag{3.9.1}$$

We remark that

$$Q_{TT} = \frac{1}{2} \left((\partial_t u)^2 + |\nabla u|^2 \right) + O(\varepsilon (\partial u)^2).$$

Proposition 3.9.1. *Let w be any of our weight functions. We have the following weighted energy estimate for u*

$$\frac{d}{dt} \left(\int Q_{TT} w(q) \right) + C \int w'(q) \left((\partial_s u)^2 + \left(\frac{\partial_\theta u}{r} \right)^2 \right) \lesssim \frac{\varepsilon}{1+t} \int w(q) (\partial u)^2 + \int w(q) |f \partial_t u|.$$

Moreover, if we use the weight modulator α defined in (3.4.22), *we obtain*

$$\frac{d}{dt} \left(\int Q_{TT} \alpha^2 w(q) \right) + C \int \alpha^2 w'(q) \left((\partial_s u)^2 + \left(\frac{\partial_\theta u}{r} \right)^2 \right) \lesssim \frac{\varepsilon}{(1+t)^{1+2\sigma}} \int w(q) (\partial u)^2 + \int \alpha^2 w(q) |f \partial_t u|.$$

Proof. We multiply (3.9.1) by $w(q)$ and integrate it on an hypersurface of constant t. We obtain

$$-\frac{d}{dt} \left(\int Q_{TT} w(q) \right) = \int w(q) \left(f \partial_t u + Q_{\alpha\beta} \pi^{\alpha\beta} \right) + \int Q_{T\alpha} D^\alpha w. \tag{3.9.2}$$

We have

$$Q_{T\alpha} D^\alpha u = -2 w'(q) g^{\alpha \underline{L}} Q_{T\alpha} = w'(q) Q_{TL} + g_{LT} w'(q) (\partial u)^2.$$

We calculate

$$\begin{aligned} Q_{TL} =& \partial_t u (\partial_t u + \partial_r u) - \frac{1}{2} (-(\partial_t u)^2 + |\nabla u|^2) + g_{LL} (\partial_q u)^2 + g_{\underline{L}\underline{L}} (\partial_s u)^2 + s.t. \\ =& \frac{1}{2} \left((\partial_s u)^2 + \left(\frac{\partial_\theta u}{r} \right)^2 \right) + g_{LL} (\partial_q u)^2 + g_{\underline{L}\underline{L}} (\partial_s u)^2 + s.t. \end{aligned}$$

where $s.t.$ denotes similar terms. Consequently, with the help of the bootstrap (3.4.16), (3.4.19) and the estimate (3.5.8) we have

$$Q_{T\alpha} D^\alpha u = \left((\partial_s u)^2 + \left(\frac{\partial_\theta u}{r} \right)^2 \right) (1 + O(\varepsilon)) w'(q) + O \left(\frac{\varepsilon w'(q)(1+|q|)}{(1+t)^{\frac{3}{2}-\rho}} (\partial u)^2 \right),$$

and since $|w'(q)| \lesssim \frac{w(q)}{1+|q|}$

$$Q_{T\alpha}D^{\alpha}u = \left((\partial_s u)^2 + \left(\frac{\partial_\theta u}{r}\right)^2\right)(1+O(\varepsilon))w'(q) + O\left(\frac{\varepsilon w(q)}{(1+t)^{\frac{3}{2}-\rho}}(\partial u)^2\right). \tag{3.9.3}$$

We now estimate the deformation tensor of T. We have

$$\pi_{\alpha\beta} = \mathcal{L}_T g_{\alpha\beta} = \partial_t g_{\alpha\beta}.$$

We obtain

$$\begin{aligned}
\pi_{LL} &= \partial_T g_{LL} = O\left(\frac{\varepsilon}{(1+t)^{\frac{3}{2}-\rho}}\right),\\
\pi_{UL} &= \partial_T g_{UL} = O\left(\frac{\varepsilon}{(1+t)^{\frac{3}{2}-\rho}}\right),\\
\pi_{L\underline{L}} &= \partial_T g_{L\underline{L}} = O\left(\frac{\varepsilon}{(1+t)^{\frac{1}{2}-\rho}(1+|q|)}\right),\\
\pi_{U\underline{L}} &= \partial_T g_{U\underline{L}} = O\left(\frac{\varepsilon}{(1+t)^{\frac{1}{2}-\rho}(1+|q|)}\right),\\
\pi_{\underline{L}\underline{L}} &= \partial_T g_{\underline{L}\underline{L}} = O\left(\frac{\varepsilon}{(1+|q|)^{\frac{3}{2}-\rho}}\right),\\
\pi_{UU} &= \partial_T g_{UU} = \frac{\partial_q(q\chi(q))b(\theta)}{r} + O\left(\frac{\varepsilon}{(1+s)(1+|q|)^{\frac{1}{2}-\rho}}\right),
\end{aligned}$$

Consequently, the terms $Q^{LL}\pi_{LL}$ and $Q^{UL}Q_{UL}$ give contributions of the form

$$\frac{\varepsilon}{(1+t)^{\frac{3}{2}-\rho}}(\partial u)^2. \tag{3.9.4}$$

We can calculate

$$Q_{L\underline{L}} = \partial_L u \partial_{\underline{L}} u - \frac{1}{2} g_{L\underline{L}}\left(2g^{L\underline{L}}\partial_L u\partial_{\underline{L}} u + (\partial_U u)^2\right) + g_{\mathcal{T}\mathcal{T}}(\partial u)^2 + s.t. = (\partial_U u)^2 + g_{\mathcal{T}\mathcal{T}}(\partial u)^2 + s.t.$$

Consequently the term $Q^{L\underline{L}}\pi_{L\underline{L}}$ gives contributions of the form

$$\frac{\varepsilon}{(1+|q|)(1+t)^{\frac{1}{2}-\rho}}(\bar{\partial} u)^2. \tag{3.9.5}$$

The terms $Q^{\underline{L}\underline{L}}\pi_{\underline{L}\underline{L}}$ and $Q^{\underline{L}U}\pi_{\underline{L}U}$ give contributions of the form

$$\frac{\varepsilon}{(1+|q|)^{\frac{3}{2}-\rho}}(\bar{\partial} u)^2, \tag{3.9.6}$$

and the term $Q^{UU}\pi_{UU}$ gives contributions of the form

$$\frac{\partial_q(q\chi(q))b(\theta)}{r}\bar{\partial} u\partial u, \quad \frac{\varepsilon}{(1+s)(1+|q|)^{\frac{1}{2}-\rho}}\bar{\partial} u\partial u. \tag{3.9.7}$$

Thanks to (3.9.2), (3.9.3), (3.9.4), (3.9.5), (3.9.6) and (3.9.7) what we obtain is

$$\begin{aligned}
&\frac{d}{dt}\left(\int Q_{TT}w(q)\right) + \frac{1}{2}\int w'(q)\left((\partial_s u)^2 + \left(\frac{\partial_\theta u}{r}\right)^2\right)\\
\lesssim &\frac{\varepsilon}{(1+t)^{\frac{3}{2}-\rho}}\int w(q)(\partial u)^2 + \varepsilon\int \frac{w(q)}{(1+|q|)^{\frac{3}{2}-\rho}}(\bar{\partial} u)^2 + \varepsilon\int w(q)\frac{\mathbb{1}_{q>1}}{r}|\partial u\bar{\partial} u| + \int w(q)|f\partial_t u|.
\end{aligned} \tag{3.9.8}$$

In the region $q>1$, we have $\frac{1}{r} \leq \frac{1}{t+1}$. Moreover, all our weight functions satisfy

$$\frac{w(q)}{(1+|q|)^{\frac{3}{2}-\rho}} \lesssim w'(q),$$

therefore, for ε small enough, we can subtract from our inequality the term

$$\varepsilon \int \frac{w(q)}{(1+|q|)^{\frac{3}{2}-\rho}} (\bar{\partial} u)^2,$$

and we obtain

$$\frac{d}{dt}\left(\int Q_{TT} w(q)\right) + C\int w'(q)\left((\partial_s u)^2 + \left(\frac{\partial_\theta u}{r}\right)^2\right) \lesssim \frac{\varepsilon}{1+t}\int w(q)(\partial u)^2 + \int w(q)|f\partial_t u|.$$

This conclude the first part of the proof of Proposition 3.9.1.

Next, we perform the estimate with the weight modulator α. If we replace w by $\alpha^2 w$ in (3.9.8), and we absorb as before the term $\varepsilon \int \frac{\alpha^2 w(q)}{(1+|q|)^{\frac{3}{2}-\rho}}(\bar{\partial} u)^2$ we obtain

$$\begin{aligned}&\frac{d}{dt}\left(\int Q_{TT}\alpha^2 w(q)\right) + \frac{1}{2}\int (\alpha^2 w)'(q)\left((\partial_s u)^2 + \left(\frac{\partial_\theta u}{r}\right)^2\right)\\ \lesssim& \frac{\varepsilon}{(1+t)^{\frac{3}{2}-\rho}}\int \alpha^2 w(q)(\partial u)^2 + \int \frac{\alpha^2 w(q)\mathbb{1}_{q>1}}{r}|\partial u \bar{\partial} u| + \int \alpha^2 w(q)|f\partial_t u|.\end{aligned}$$

We write

$$\frac{\mathbb{1}_{q>1}}{r} \leq \frac{\mathbb{1}_{q>1}(1+|q|)^\sigma}{(1+t)^{\frac{1}{2}+\sigma}(1+|q|)^{\frac{1}{2}}},$$

and so we estimate, since in the region $q>1$ we have $\alpha(q) = (1+|q|)^{-\sigma}$

$$\begin{aligned}\varepsilon\int \frac{\alpha^2(q)w(q)\mathbb{1}_{q>1}}{r}|\partial u\bar{\partial} u| &\leq \varepsilon \int \frac{\alpha(q)w(q)\mathbb{1}_{q>1}}{(1+t)^{\frac{1}{2}+\sigma}(1+|q|)^{\frac{1}{2}}}|\partial u \bar{\partial} u|\\ &\leq \frac{\varepsilon}{t^{1+2\sigma}}\int \mathbb{1}_{q>1} w(q)(\partial u)^2 + \varepsilon\int \mathbb{1}_{q>1}\frac{\alpha^2(q)w(q)}{1+|q|}(\bar{\partial} u)^2.\end{aligned}$$

Moreover $\mathbb{1}_{q>1}\frac{\alpha^2(q)w(q)}{1+|q|} \lesssim (\alpha^2 w)'$. Therefore

$$\begin{aligned}&\frac{d}{dt}\left(\int Q_{TT}\alpha^2 w(q)\right) + C\int (\alpha^2 w)'(q)\left((\partial_s u)^2 + \left(\frac{\partial_\theta u}{r}\right)^2\right)\\ \lesssim& \frac{\varepsilon}{(1+t)^{\frac{3}{2}-\rho}}\int \alpha^2 w(q)(\partial u)^2 + \frac{\varepsilon}{t^{1+2\sigma}}\int \mathbb{1}_{q>1}w(q)(\partial u)^2 + \varepsilon\int (w\alpha^2)'(q)(\bar{\partial} u)^2 + \int \alpha^2 w(q)|f\partial_t u|.\end{aligned}$$

We note that with our weight functions and the definition of α, we have $\alpha^2 w' \sim (\alpha^2 w)'$. For ε small enough, we can absorb the term

$$\varepsilon \int w'(q)\alpha^2(q)(\bar{\partial} u)^2$$

to obtain

$$\begin{aligned}&\frac{d}{dt}\left(\int Q_{TT}\alpha^2 w(q)\right) + C\int \alpha^2 w'(q)\left((\partial_s u)^2 + \left(\frac{\partial_\theta u}{r}\right)^2\right)\\ \lesssim& \frac{\varepsilon}{(1+t)^{\frac{3}{2}-\rho}}\int \alpha^2 w(q)(\partial u)^2 + \frac{\varepsilon}{t^{1+2\sigma}}\int \mathbb{1}_{q>1}w(q)(\partial u)^2 + \int \alpha^2 w(q)|f\partial_t u|,\end{aligned}$$

which concludes the proof of Proposition 3.9.1. $\square$

3.10 Commutation with the vector fields and L^2 estimate

3.10.1 Estimation for $I \leq N$

We note for $J < N$

$$E_J = \sum_{I \leq J} \|w_0(q)^{\frac{1}{2}} \partial Z^I \varphi\|_{L^2}^2 + \|w_2(q)^{\frac{1}{2}} \partial Z^I \widetilde{g}_3\|_{L^2}^2 + \frac{1}{\varepsilon(1+t)} \|w_3(q)^{\frac{1}{2}} \partial Z^I h\|_{L^2}^2$$

and

$$\begin{aligned} E_N = \sum_{I \leq N} & \|\alpha_2 w_0(q)^{\frac{1}{2}} \partial Z^I \varphi\|_{L^2}^2 + \|\alpha_2 w_2(q)^{\frac{1}{2}} \partial Z^I \widetilde{g}_4\|_{L^2}^2 \\ & + \frac{1}{\varepsilon(1+t)} \|\alpha_2 w_3(q)^{\frac{1}{2}} \partial Z^I h\|_{L^2}^2 + \frac{1}{\varepsilon(1+t)} \|\alpha_2 w_3(q)^{\frac{1}{2}} \partial Z^I k\|_{L^2}^2. \end{aligned}$$

We also note for $J < N$

$$A_J = \sum_{I \leq J} \|w_0'(q)^{\frac{1}{2}} \bar{\partial} Z^I \varphi\|_{L^2}^2 + + \|w_2'(q)^{\frac{1}{2}} \bar{\partial} Z^I \widetilde{g}_3\|_{L^2}^2 + \frac{1}{\varepsilon(1+t)} \|w_3'(q)^{\frac{1}{2}} \bar{\partial} Z^I h\|_{L^2}^2$$

and

$$\begin{aligned} A_N = \sum_{I \leq J} & \|\alpha_2 w_0'(q)^{\frac{1}{2}} \bar{\partial} Z^I \varphi\|_{L^2}^2 + + \|\alpha_2 w_2'(q)^{\frac{1}{2}} \bar{\partial} Z^I \widetilde{g}_4\|_{L^2}^2 \\ & + \frac{1}{\varepsilon(1+t)} \|\alpha_2 w_3'(q)^{\frac{1}{2}} \bar{\partial} Z^I h\|_{L^2}^2 + \frac{1}{\varepsilon(1+t)} \|\alpha_2 w_3'(q)^{\frac{1}{2}} \bar{\partial} Z^I k\|_{L^2}^2. \end{aligned}$$

Remark 3.10.1. *Because of the decompositions* (3.4.8) *and* (3.4.9) *for the metric, and the non commutation of the wave operator with the null decomposition, we have to deal with terms of the form* $\frac{\partial_\theta h}{r^2}$ *in the equation for* $\widetilde{g}_4$ *or* $\widetilde{g}_3$*. Written like this, these terms are not quadratic. However, since we choose for* h *zero initial data, and since the equation for* h *is quadratic,* h *in itself is quadratic. To carry this information along the proof, we may divide in the energies* E_I *the norms involving* h *and* k *by* ε*. Since the initial data for* h *and* k *are zero, we have*

$$E_I(0) \leq C_0^2 \varepsilon^2. \tag{3.10.1}$$

Proposition 3.10.2. *We have the estimates for* $I \leq N$*,*

$$E_I \leq (C_0^2 \varepsilon^2 + \varepsilon^2)(1+t)^{C\sqrt{\varepsilon}},$$

and for $\kappa \gg \varepsilon$

$$\int_0^t \frac{1}{(1+t)^\kappa} A_I \lesssim \varepsilon^2.$$

This is a straightforward consequence of the following proposition.

Proposition 3.10.3. *We have the inequality, up to some negligible terms defined in Lemmas 3.10.4, 3.10.5 and 3.10.6 for* $I \leq N$

$$\frac{d}{dt} E_I + A_I \lesssim \frac{\sqrt{\varepsilon}}{1+t} E_I + \frac{\varepsilon^{\frac{5}{2}}}{1+t}.$$

We first prove Proposition 3.10.2, admitting Proposition 3.10.3.

Proof of Proposition 3.10.2. We have proved

$$\frac{d}{dt}E_I \leq C\frac{\sqrt{\varepsilon}}{1+t}E_I + C\frac{\varepsilon^{\frac{5}{2}}}{1+t},$$

therefore, if we note $E_I = G(1+t)^{C\sqrt{\varepsilon}}$, we have

$$\frac{d}{dt}G \leq C\frac{\varepsilon^{\frac{5}{2}}}{(1+t)^{1+C\sqrt{\varepsilon}}}.$$

After integrating, we obtain

$$G(t) \leq G(0) + \varepsilon^2 - \frac{\varepsilon^2}{(1+t)^{C\sqrt{\varepsilon}}},$$

and hence

$$E_I \leq (E_I(0)+\varepsilon^2)(1+t)^{C\sqrt{\varepsilon}} \leq (C_0^2\varepsilon^2+\varepsilon^2)(1+t)^{C\sqrt{\varepsilon}}.$$

Moreover, we have

$$\frac{d}{dt}E_I + A_I \leq C\frac{\sqrt{\varepsilon}}{1+t}E_I + \frac{\varepsilon^{\frac{5}{2}}}{1+t},$$

therefore if we multiply this inequality by $\frac{1}{(1+t)^\kappa}$ we obtain

$$\frac{d}{dt}\left(\frac{E_I}{(1+t)^\kappa}\right) + \frac{A_I}{(1+t)^\kappa} \leq \frac{1}{(1+t)^\kappa}\left(\frac{d}{dt}E_I + A_I\right) \leq \frac{C\sqrt{\varepsilon}}{(1+t)^{1+\kappa}}E_I + \frac{C\varepsilon^{\frac{5}{2}}}{(1+t)^{1+\kappa}} \leq \frac{C\varepsilon^{\frac{5}{2}}}{(1+t)^{1+\kappa-C\sqrt{\varepsilon}}}.$$

Therefore, if $C\sqrt{\varepsilon} < \kappa$, the right-hand side is integrable and so

$$\int \frac{1}{(1+t)^\kappa}A_I \lesssim \varepsilon^2.$$

This concludes the proof of Proposition 3.10.2. $\square$

Proposition 3.10.3 is a direct consequence of the three following lemmas.

Lemma 3.10.4. *We have the inequality,*

$$\frac{d}{dt}\|\alpha_2 w(q)^{\frac{1}{2}}\partial\widetilde{Z^N\varphi}\|_{L^2}^2 + \|\alpha_2 w'(q)^{\frac{1}{2}}\bar{\partial}\widetilde{Z^N\varphi}\|_{L^2}^2 \lesssim \frac{\varepsilon}{1+t}E_N + \varepsilon\|\alpha_2 w_2'(q)^{\frac{1}{2}}\bar{\partial}Z^N\widetilde{g}_4\|_{L^2}^2 + \frac{\varepsilon^3}{1+t}$$

where $\widetilde{Z^N\varphi} - Z^N\varphi$ is composed of terms of the form

$$\frac{\chi(q)q\partial\varphi\partial_\theta^{N-1}b}{g_{UU}},$$

and we have

$$\|\alpha_2 w_0^{\frac{1}{2}}\partial(\widetilde{Z^N\varphi} - Z^N\varphi)\|_{L^2} \lesssim \varepsilon^2.$$

For $I<N$ we have

$$\frac{d}{dt}\|w_0(q)^{\frac{1}{2}}\partial Z^I\varphi\|_{L^2}^2 + \|w_0'(q)^{\frac{1}{2}}\bar{\partial}Z^I\varphi\|_{L^2}^2 \lesssim \frac{\varepsilon}{1+t}E_I + \varepsilon\|w_2'(q)^{\frac{1}{2}}\bar{\partial}Z^I\widetilde{g}_4\|_{L^2}^2.$$

Lemma 3.10.5. *We have the inequality,*

$$\frac{d}{dt}\left(\frac{1}{\varepsilon t}\|\alpha_2 w_3(q)^{\frac{1}{2}}\partial\widetilde{Z^N h}\|_{L^2}^2\right) + \frac{1}{\varepsilon(1+t)}\|\alpha_2 w_3'(q)^{\frac{1}{2}}\bar{\partial}\widetilde{Z^N h}\|_{L^2}^2$$
$$\lesssim \frac{\sqrt{\varepsilon}}{1+t}E_N + \sqrt{\varepsilon}\|\alpha_2 w_2'^{\frac{1}{2}}(q)\bar{\partial}\widetilde{Z}^N g_4\|_{L^2}^2 + \frac{\varepsilon^{\frac{5}{2}}}{1+t},$$

where $\widetilde{Z^N h} - Z^N h$ *is composed of terms of the form*

$$\frac{\chi(q) q \partial h \partial_\theta^{N-1} b}{g_{UU}},$$

and we have

$$\|\alpha_2 w_0^{\frac{1}{2}} \partial(\widetilde{Z^N h} - Z^N h)\|_{L^2} \lesssim \varepsilon^3 \sqrt{1+t}.$$

We have a similar estimate for k

$$\begin{aligned} &\frac{d}{dt}\left(\frac{1}{\varepsilon t}\|\alpha_2 w_3(q)^{\frac{1}{2}} \partial Z^N k\|_{L^2}^2\right) + \frac{1}{\varepsilon(1+t)}\|\alpha_2 w_3'(q)^{\frac{1}{2}} \bar{\partial} Z^N k\|_{L^2}^2 \\ \lesssim &\frac{\sqrt{\varepsilon}}{1+t} E_N + \sqrt{\varepsilon}\|\alpha_2 w_2'^{\frac{1}{2}}(q) \bar{\partial} \widetilde{Z}^I g_4\|_{L^2}^2. \end{aligned}$$

Moreover for $I < N$

$$\begin{aligned} &\frac{d}{dt}\left(\frac{1}{\varepsilon t}\|w_3(q)^{\frac{1}{2}} \partial Z^I h\|_{L^2}^2\right) + \frac{1}{\varepsilon(1+t)}\|w_3'(q)^{\frac{1}{2}} \bar{\partial} Z^I h\|_{L^2}^2 \\ \lesssim &\frac{\sqrt{\varepsilon}}{1+t} E_I + \sqrt{\varepsilon}\|w_2'^{\frac{1}{2}}(q) \bar{\partial} \widetilde{Z}^I g_4\|_{L^2}^2 + \frac{\varepsilon^{\frac{5}{2}}}{1+t}, \end{aligned}$$

Lemma 3.10.6. *We have the estimate*

$$\begin{aligned} &\frac{d}{dt}\|\alpha_2 w_2(q)^{\frac{1}{2}} \partial \widetilde{Z^N \widetilde{g}_4}\|_{L^2}^2 + \|\alpha_2 w_2'(q)^{\frac{1}{2}} \partial \widetilde{Z^N \widetilde{g}_4}\|_{L^2}^2 \\ \lesssim &\frac{\sqrt{\varepsilon}}{1+t} E_N + \sqrt{\varepsilon}\frac{1}{\varepsilon(1+t)}(\|\alpha_2 w_3'(q)^{\frac{1}{2}} \bar{\partial} Z^N h\|_{L^2}^2 + \|\alpha_2 w_3'(q)^{\frac{1}{2}} \bar{\partial} Z^N k\|_{L^2}^2) \end{aligned}$$

where $\widetilde{Z^N \widetilde{g}_4} - Z^N \widetilde{g}_4$ *is composed of terms of the form*

$$-\frac{\chi(q) q \partial \widetilde{g}_4 \partial_\theta^{N-1} b}{g_{UU}}, \quad h Z^N g_{LL} dq ds$$

and we have

$$\|\alpha_2 w_0^{\frac{1}{2}} \partial(\widetilde{Z^N \widetilde{g}_4} - Z^N \widetilde{g}_4)\|_{L^2} \lesssim \varepsilon^2 + \varepsilon\|\alpha_2 w_2^{\frac{1}{2}} Z^N \widetilde{g}_{LL}\|_{L^2}.$$

For $I < N$*, we have*

$$\frac{d}{dt}\|w_2(q)^{\frac{1}{2}} \partial Z^I \widetilde{g}_3\|_{L^2}^2 + \|w_2'(q)^{\frac{1}{2}} \partial Z^I \widetilde{g}_3\|_{L^2}^2 \lesssim \frac{\sqrt{\varepsilon}}{1+t} E_I + \sqrt{\varepsilon}\frac{1}{\varepsilon(1+t)}\|w_3'(q)^{\frac{1}{2}} \bar{\partial} Z^I h\|_{L^2}^2.$$

We prove Proposition 3.10.3.

Proof of Proposition 3.10.3. Therefore, if we combine Lemmas 3.10.4, 3.10.5 and 3.10.6 we obtain

$$\frac{d}{dt} E_I + A_I \lesssim \frac{\sqrt{\varepsilon}}{1+t} E_I + \sqrt{\varepsilon} A_I + \frac{\varepsilon^{\frac{5}{2}}}{1+t},$$

and therefore

$$\frac{d}{dt} E_I + (1 - C\sqrt{\varepsilon}) A_I \lesssim \frac{\sqrt{\varepsilon}}{1+t} E_I + \frac{\varepsilon^{\frac{5}{2}}}{1+t}.$$

If ε is small enough, we have $1 - C\sqrt{\varepsilon} \geq \frac{1}{2}$, which concludes the proof of Proposition 3.10.3. □

It is sufficient to prove these three lemmas for $I = N$. For $I < N$ everything work in the same way. The weight modulator α_2 is only needed to estimate a particular term for $I = N$ and is no longer needed for $I < N$.

Proof of Lemma 3.10.4. We start with the estimates for φ. We use the weighted energy estimate for the equation

$$\Box_g Z^N\varphi = \sum_{\substack{I+J\leq N\\ J\leq N-1}}\left(Z^I g^{\alpha\beta}\right)\left(Z^J\partial_\alpha\partial_\beta\varphi\right)+\sum_{\substack{I+J\leq N\\ J\leq N-1}} Z^I H_b^\rho Z^J\partial_\rho\varphi. \tag{3.10.2}$$

It yields

$$\begin{aligned}&\frac{d}{dt}\left(\|\alpha_2 w_0(q)^{\frac12}\partial Z^N\varphi\|^2_{L^2}\right)+\|\alpha_2 w_0'(q)^{\frac12}\bar\partial Z^N\varphi\|^2_{L^2}\\ &\lesssim \left\|\alpha_2 w_0\Box_g Z^N\varphi\right\|_{L^2}\|\alpha_2 w_0(q)^{\frac12}\partial Z^N\varphi\|_{L^2}+\frac{\varepsilon}{1+t}\|\alpha_2 w_0(q)^{\frac12}\partial Z^N\varphi\|^2_{L^2}.\end{aligned}$$

Estimate of the first term Thanks to Remark 3.2.2, it is sufficient to estimate

$$\left|\sum_{\substack{I+J\leq N\\ J\leq N-1}} Z^I g_{LL}\partial_q^2 Z^J\varphi\right|\lesssim\frac{1}{(1+|q|)}\sum_{\substack{I+J\leq N\\ J\leq N-1}}|Z^I g_{LL}\partial_q Z^{J+1}\varphi|.$$

If $I\leq\frac N2\leq N-15$, we can estimate thanks to (3.5.8)

$$|Z^I g_{LL}|\lesssim\frac{\varepsilon(1+|q|)}{(1+t)^{\frac32-\rho}}$$

so

$$\left\|\frac{\alpha_2 w_0^{\frac12}}{(1+|q|)}Z^I g_{LL}\partial_q Z^J\varphi\right\|_{L^2}\lesssim\frac{\varepsilon}{(1+t)^{\frac32-\rho}}\|\alpha_2 w_0^{\frac12}\partial_q Z^J\varphi\|_{L^2}. \tag{3.10.3}$$

If $J\leq\frac N2$, we can estimate

$$|\partial_q\varphi|\lesssim\frac{\varepsilon}{(1+|q|)^{\frac32-4\rho}\sqrt{1+t}}.$$

Therefore,

$$\left\|\frac{\alpha_2 w_0^{\frac12}}{(1+|q|)}Z^I g_{LL}\partial_q Z^J\varphi\right\|_{L^2}\lesssim\frac{\varepsilon}{\sqrt{1+t}}\left\|\frac{\alpha_2 w_0(q)^{\frac12}}{(1+|q|)^{\frac52-4\rho}}Z^I g_{LL}\right\|_{L^2}\lesssim\frac{\varepsilon}{\sqrt{1+t}}\left\|\frac{\alpha_2 v(q)^{\frac12}}{1+|q|}Z^I g_{LL}\right\|_{L^2},$$

where

$$\begin{cases} v(q)=\frac{1}{(1+|q|)^{\frac52-4\rho}}\ for\ q<0,\\ v(q)=\frac{w_0(q)}{(1+|q|)}=(1+|q|)^{1+2\delta}\ for\ q>0.\end{cases}$$

We do not keep all the decay in q in the region $q>0$ in order to be in the range of application of the weighted Hardy inequality and we obtain

$$\left\|\frac{\alpha_2 w_0^{\frac12}}{(1+|q|)}Z^I g_{LL}\partial_q Z^J\varphi\right\|_{L^2}\lesssim\frac{\varepsilon}{\sqrt{1+t}}\|\alpha_2 v(q)^{\frac12}\partial_q Z^I g_{LL}\|_{L^2}.$$

We use Proposition 3.5.1, which gives

$$\partial_q Z^N g_{LL}\sim\bar\partial Z^N(\widetilde g_{L\underline{L}}+\widetilde g_{\mathcal{T}\mathcal{T}}). \tag{3.10.4}$$

Consequently, thanks to Remark 3.4.10, we have $\partial_q Z^N g_{LL}\sim\bar\partial Z^N\widetilde g_4$. Moreover, we calculate

$$\begin{cases} w_2'(q)=\frac{1+2\mu}{(1+|q|)^{2+2\mu}}\ for\ q<0,\\ w_2'(q)=(2+2\delta)(1+|q|)^{1+2\delta}\ for\ q>0.\end{cases}$$

Therefore, $v \lesssim w_2'$ and we obtain

$$\left\| \frac{\alpha_2 w_0^{\frac{1}{2}}}{(1+|q|)} Z^I g_{LL} \partial_q Z^J \varphi \right\|_{L^2} \|\alpha_2 w_0(q)^{\frac{1}{2}} \partial Z^N \varphi\|_{L^2} \lesssim \frac{\varepsilon}{1+t} \|\alpha_2 w_0(q)^{\frac{1}{2}} \partial Z^N \varphi\|_{L^2}^2 + \varepsilon \left\| \alpha_2 w_2'(q)^{\frac{1}{2}} \bar{\partial} \widetilde{Z}^N g_4 \right\|_{L^2}^2 . \tag{3.10.5}$$

Estimate of the second term The second term contains only the crossed term, which occur only in the region $q > 0$. Thanks to the discussion of Section 3.2.5, it is sufficient to estimate (3.2.13), which gives a contribution of the form

$$Z^N \left(\partial (g_b)_{UU} \partial \varphi \right).$$

For $I \leq N-2$ we have

$$|Z^I \partial (g_b)_{UU}| \lesssim \frac{\varepsilon \mathbb{1}_{q>0}}{r}$$

and consequently

$$\|\alpha_2 w_0^{\frac{1}{2}} \partial Z^I (g_b)_{UU} \partial Z^{N-I} \varphi\|_{L^2} \lesssim \frac{\varepsilon}{1+t} \|\alpha_2 w^{\frac{1}{2}} \partial Z^{N-I} \varphi\|_{L^2}. \tag{3.10.6}$$

In $\partial Z^I (g_b)_{UU} \partial_\rho Z^{N-I} \varphi$ with $I \geq N-2$, we have to note the presence of terms of the form

$$\frac{\chi(q) q \partial_\theta^{N+1} b(\theta)}{r^2} \partial \varphi, \tag{3.10.7}$$

which require a special treatment since $\partial_\theta^{N+1} b(\theta)$ does not belong to L^2. To deal with these terms we write

$$\left| \Box_g \left(\frac{\chi(q) q \partial_\theta^{N-1} b}{g_{UU}} \partial \varphi \right) - \frac{\chi(q) q \partial_\theta^{N+1} b(\theta)}{r^2} \partial \varphi \right| \lesssim \frac{\chi(q)}{1+s} \left(\sum_{I \leq 2} |\partial Z^I \varphi| \right) \left(|\partial_\theta^N b| + |\partial_\theta^{N-1} b| \right) + s.t.$$

We can estimate, thanks to the estimate (3.4.31) for $\partial \varphi$,

$$\left\| w_0^{\frac{1}{2}} \partial \left(\chi(q) q \partial \varphi \partial_\theta^{N-1} b \right) \right\|_{L^2} \lesssim \left\| \frac{\varepsilon}{\sqrt{1+s}(1+|q|)^{\frac{1}{2}+2\sigma-\sigma}} \partial_\theta^N b \right\|_{L^2} \lesssim \varepsilon^2. \tag{3.10.8}$$

Therefore, we may perform the energy estimate for $\widetilde{Z^N \varphi} = Z^N \varphi - \frac{\chi(q) q \partial \varphi \partial_\theta^{N-1} b}{g_{UU}}$ instead of $Z^N \varphi$. We are reduced to estimate

$$\begin{aligned} \left\| \alpha_2 w_0^{\frac{1}{2}} \frac{\chi(q)}{1+s} \left(\sum_{I \leq 2} |\partial Z^I \varphi| \right) \left(|\partial_\theta^N b| + |\partial_\theta^{N-1} b| \right) \right\|_{L^2} &\lesssim \left\| \frac{\varepsilon}{(1+s)^{\frac{3}{2}} (1+|q|)^{\frac{1}{2}+\sigma}} \left(|\partial_\theta^N b| + |\partial_\theta^{N-1} b| \right) \right\|_{L^2} \\ &\lesssim \frac{\varepsilon^3}{1+t}. \end{aligned} \tag{3.10.9}$$

The other terms in $\partial Z^I (g_b)_{UU} \partial Z^{N-I} \varphi$ with $I \geq N-2$, give contributions similar to (3.10.9).

Remark 3.10.7. *We introduce the weight modulator α_2 to deal with the term* (3.10.7) *which is only present for $I = N$. It is no longer needed for $I < N$. To see this, let us estimate $\frac{\chi(q) q \partial_\theta^N b(\theta)}{r^2} \partial \varphi$ which is the analogue of* (3.10.7) *for $I = N-1$.*

$$\left\| w_0^{\frac{1}{2}} \frac{\chi(q) q \partial_\theta^N b(\theta)}{r^2} \partial \varphi \right\|_{L^2} \lesssim \sum_{I \leq 2} \|w_0^{\frac{1}{2}} \partial Z^I \varphi\|_{L^2} \left\| \frac{\chi(q) q \partial_\theta^N b}{r^2 \sqrt{1+s} \sqrt{1+|q|}} \right\|_{L^2} \lesssim \frac{1}{1+t} \sqrt{E_2} \|\partial_\theta^N b\|_{L^2(\mathbb{S}^1)},$$

where we have used the weighted Klainerman-Sobolev inequality

$$|w_0^{\frac{1}{2}}\partial\varphi| \lesssim \frac{1}{\sqrt{1+s}\sqrt{1+|q|}}\sum_{I\leq 2}\|w_0^{\frac{1}{2}}\partial Z^I\varphi\|_{L^2},$$

and consequently

$$\left\| w_0^{\frac{1}{2}}\frac{\chi(q)q\partial_\theta^N b(\theta)}{r^2}\partial\varphi\right\|_{L^2} \lesssim \frac{\varepsilon^2}{1+t}\sqrt{E_2}. \tag{3.10.10}$$

Thanks to (3.10.3), (3.10.5), (3.10.6), (3.10.9) we obtain

$$\frac{d}{dt}\left(\|\alpha_2 w_0(q)^{\frac{1}{2}}\partial\widetilde{Z^N\varphi}\|_{L^2}^2\right) + \|\alpha w'(q)^{\frac{1}{2}}\bar\partial\widetilde{Z^N\varphi}\|_{L^2}^2 \lesssim \frac{\varepsilon}{1+t}E_N + \varepsilon\|\alpha_2 w_2'(q)^{\frac{1}{2}}\bar\partial\widetilde{g}_4\|_{L^2}^2 + \frac{\varepsilon^3}{1+t}, \tag{3.10.11}$$

which, with the estimate (3.10.8) for $\widetilde{Z^N\varphi} - Z^N\varphi$ concludes the proof of Lemma 3.10.4. □

Proof of Lemma 3.10.5. We now estimate h. The equation for $Z^N h$ writes

$$\begin{aligned}\Box_g Z^I h = &\sum_{\substack{I+J\leq N\\ J\leq N-1}}\left(Z^I g^{\alpha\beta}\right)\left(Z^J\partial_\alpha\partial_\beta h\right) + \sum_{\substack{I+J\leq N\\ J\leq N-1}} Z^I H^\rho Z^J\partial_\rho h\\ &+ Z^I((\partial_q\varphi)^2 + (R_b)_{qq} + Q_{\underline{LL}}(h,\widetilde{g})).\end{aligned} \tag{3.10.12}$$

Estimate of the first term Following Remark 3.2.2, it is sufficient to estimate $Z^I g_{LL}\partial_q^2 Z^J h$. For $I\leq\frac{N}{2}$, similarly than (3.10.3) we have

$$\left\|\frac{\alpha_2 w_3(q)^{\frac{1}{2}}}{(1+|q|)}Z^I g_{LL}\partial_q Z^J h\right\|_{L^2} \lesssim \frac{\varepsilon}{(1+t)^{\frac{3}{2}-\rho}}\|\alpha_2 w_3(q)^{\frac{1}{2}}\partial_q Z^J h\|_{L^2}. \tag{3.10.13}$$

For $J\leq\frac{N}{2}$, we have the estimate, thanks to (3.4.42),

$$|\partial_q Z^J h| \lesssim \frac{\varepsilon}{(1+|q|)^{\frac{3}{2}-\rho}},$$

so

$$\left\|\frac{\alpha_2 w_3(q)^{\frac{1}{2}}}{(1+|q|)}Z^I g_{LL}\partial_q Z^J h\right\|_{L^2} \lesssim \left\|\frac{\alpha_2}{1+|q|}\left(\frac{w_3(q)}{(1+|q|)^{3-2\rho}}\right)^{\frac{1}{2}}Z^I g_{LL}\right\|_{L^2}.$$

We have

$$\frac{w_3(q)}{(1+|q|)^{3-2\rho}} \leq \begin{cases}\frac{1}{(1+|q|)^{3-2\rho}} & for\ q<0,\\ (1+|q|)^{2\delta-2\rho}\leq(1+|q|)^{1+2\delta} & for\ q>0.\end{cases}$$

This yields

$$\frac{w_3(q)}{(1+|q|)^{3-2\rho}} \lesssim w_2'(q).$$

Therefore the weighted Hardy inequality and the wave coordinate condition give, similarly than for (3.10.5),

$$\left\|\frac{\alpha_2 w_3(q)^{\frac{1}{2}}}{(1+|q|)}Z^I g_{LL}\partial_q Z^J h\right\|_{L^2} \lesssim \varepsilon\|\alpha_2 w_2'(q)^{\frac{1}{2}}\partial_q Z^I g_{LL}\|_{L^2} \lesssim \varepsilon\|\alpha_2 w_2'^{\frac{1}{2}}(q)\bar\partial\widetilde{g}_4\|_{L^2}. \tag{3.10.14}$$

Estimate of the second term The second term contains crossed terms, which can be studied exactly in the same way than for φ. Similarly than (3.10.6), we have for $I \leq N-2$

$$\|\alpha_2 w_3^{\frac{1}{2}} \partial Z^I (g_b)_{UU} \partial Z^{N-I} h\|_{L^2} \lesssim \frac{\varepsilon}{1+t} \|\alpha_2 w^{\frac{1}{2}} \partial Z^{N-I} h\|_{L^2}. \tag{3.10.15}$$

Like for φ the following term require a special treatment.

$$\frac{\chi(q) q \partial_\theta^{N+1} b(\theta)}{r^2} \partial h. \tag{3.10.16}$$

We have

$$\left| \Box_g \left(\frac{\chi(q) q \partial\varphi \partial_\theta^{N-1} b}{g_{UU}} \partial h \right) - \frac{\chi(q) q \partial_\theta^{N+1} b(\theta)}{r^2} \partial h \right| \lesssim \frac{\chi(q)}{1+s} \left(\sum_{I \leq 2} |\partial Z^I h| \right) \left(|\partial_\theta^N b| + |\partial_\theta^{N-1} b| \right) + s.t.$$

We can estimate, thanks to the estimate (3.4.33) for ∂h,

$$\left\| \alpha_2 w_3^{\frac{1}{2}} \partial \left(\chi(q) q \partial h \partial_\theta^{N-1} b \right) \right\|_{L^2} \lesssim \left\| \frac{\varepsilon}{(1+|q|)^{1+2\sigma-\sigma}} \partial_\theta^N b \right\|_{L^2} \lesssim \varepsilon^2 \sqrt{1+t}. \tag{3.10.17}$$

Therefore, we may perform the energy estimate for $\widetilde{Z^N h} = Z^N h - \frac{\chi(q) q \partial h \partial_\theta^{N-1} b}{g_{UU}}$ instead of $Z^N h$. We are reduced to estimate

$$\begin{aligned} \left\| \alpha_2 w_3^{\frac{1}{2}} \frac{\chi(q)}{1+s} \left(\sum_{I \leq 2} |\partial Z^I h| \right) \left(|\partial_\theta^N b| + |\partial_\theta^{N-1} b| \right) \right\|_{L^2} &\lesssim \left\| \frac{\varepsilon}{(1+s)(1+|q|)^{1+\sigma}} \left(|\partial_\theta^N b| + |\partial_\theta^{N-1} b| \right) \right\|_{L^2} \\ &\lesssim \frac{\varepsilon^3}{\sqrt{1+t}}. \end{aligned} \tag{3.10.18}$$

The other terms in $\partial Z^I (g_b)_{UU} \partial Z^{N-I} h$ with $I \geq N-2$, give contributions similar to (3.10.18).

Estimate of $Z^N (\partial_q \varphi)^2$ We have

$$\|\alpha_2 w_3(q)^{\frac{1}{2}} Z^N \left((\partial_q \varphi)^2 \right) \|_{L^2} \lesssim \sum_{I+J \leq N} \|\alpha_2 w_3(q)^{\frac{1}{2}} \partial_q Z^I \varphi \partial_q Z^J \varphi\|_{L^2}.$$

We can assume $I \leq \frac{N}{2}$ and estimate thanks to (3.4.28)

$$|\partial_q Z^I \varphi| \lesssim \frac{\varepsilon}{\sqrt{1+|q|}\sqrt{1+t}}.$$

Then, since

$$\frac{w_3^{\frac{1}{2}}}{\sqrt{1+|q|}} \leq w_0^{\frac{1}{2}},$$

we obtain

$$\|\alpha_2 w_3(q)^{\frac{1}{2}} Z^N (\partial_q \varphi)^2\|_{L^2} \lesssim \frac{\varepsilon}{\sqrt{1+t}} \sum_{I \leq N} \|\alpha_2 w_0(q)^{\frac{1}{2}} \partial_q Z^J \varphi\|_{L^2}. \tag{3.10.19}$$

Estimate of $Z^N(R_b)_{qq}$ Thanks to (3.1.7), the main contribution in $(R_b)_{qq}$ is

$$\frac{\partial_q^2(q\chi(q))b(\theta)}{r},$$

which is supported in $1 \leq q \leq 2$. We estimate

$$\left\| \alpha_2 w_3(q)^{\frac{1}{2}} Z^N \left(\frac{b(\theta)\partial_q^2(q\chi(q))}{r} \right) \right\|_{L^2} \lesssim \frac{1}{\sqrt{1+t}} \sum_{I \leq N} \|\partial_\theta^I b\|_{L^2(\mathbb{S}^1)} \lesssim \frac{\varepsilon^2}{\sqrt{1+t}}. \tag{3.10.20}$$

Estimate of $Z^N Q_{\underline{L}\underline{L}}(h, \widetilde{g})$ We recall from (3.2.12) that

$$Q_{\underline{L}\underline{L}}(h, \widetilde{g}) = \partial_{\underline{L}} g_{LL} \partial_{\underline{L}} h + \partial_{\underline{L}} g_{L\underline{L}} \partial_L h + \partial_{\underline{L}}(g_b)_{UU} \partial_{\underline{L}} g_{\underline{L}L}.$$

The terms $Z^N(\partial_{\underline{L}} g_{LL} \partial_{\underline{L}} h)$ and $Z^N(\partial_{\underline{L}} g_{L\underline{L}} \partial_L h)$ may be treated in a similar way than the quasilinear term, giving contributions similar to (3.10.13) and (3.10.14). The term $Z^N(\partial_{\underline{L}}(g_b)_{UU} \partial_{\underline{L}} g_{\underline{L}L})$ is a crossed term, hence it is supported only in the region $q > 0$. It is sufficient to estimate $\partial_{\underline{L}}(g_b)_{UU} \partial_{\underline{L}} Z^N g_{\underline{L}L}$. We have

$$|\partial_q(g_b)_{UU}| \lesssim \frac{\varepsilon \mathbb{1}_{q>0}}{r} \lesssim \frac{\varepsilon \mathbb{1}_{q>0}}{\sqrt{1+t}\sqrt{1+|q|}}.$$

so we can estimate

$$\|\alpha_2 w_3^{\frac{1}{2}} \partial_q(g_b)_{UU} \partial Z^N g_{L\underline{L}}\|_{L^2} \lesssim \frac{\varepsilon}{\sqrt{1+t}} \left\| \frac{\alpha_2 w_3^{\frac{1}{2}} \mathbb{1}_{q>0}}{\sqrt{1+|q|}} \partial Z^N g_{L\underline{L}} \right\|_{L^2} \lesssim \frac{\varepsilon}{\sqrt{1+t}} \left\| \alpha_2 w_2^{\frac{1}{2}} \partial Z^N g_{L\underline{L}} \right\|_{L^2},$$

and consequently, since $\widetilde{g}_{L\underline{L}} = (\widetilde{g}_4)_{L\underline{L}}$ we have

$$\|\alpha_2 w_3^{\frac{1}{2}} \partial_q(g_b)_{UU} \partial Z^N g_{L\underline{L}}\|_{L^2} \lesssim \frac{\varepsilon}{\sqrt{1+t}} \left\| \alpha_2 w_2^{\frac{1}{2}} \partial Z^N \widetilde{g}_4 \right\|_{L^2}. \tag{3.10.21}$$

In view of (3.10.13), (3.10.14), (3.10.15), (3.10.18), (3.10.19), (3.10.20), (3.10.21), the energy inequality yields

$$\begin{aligned}
&\frac{d}{dt}\|\alpha_2 w_3(q)^{\frac{1}{2}} \partial \widetilde{Z^N h}\|_{L^2}^2 + \|\alpha_2 w_3'(q)^{\frac{1}{2}} \bar{\partial} \widetilde{Z^N h}\|_{L^2}^2 \\
&\lesssim \Bigg(\frac{\varepsilon}{1+t}\|\alpha_2 w_3^{\frac{1}{2}} \partial Z^N h\|_{L^2} + \varepsilon \|\alpha_2 w_2'(q)^{\frac{1}{2}} \bar{\partial} Z^N \widetilde{g}_4\|_{L^2} \\
&+ \frac{\varepsilon}{\sqrt{1+t}} \left(\|\alpha_2 w_0^{\frac{1}{2}} \partial_q Z^J \varphi\|_{L^2} + \|\alpha_2 w_2^{\frac{1}{2}} \partial_q Z^J \widetilde{g}_4\|_{L^2} \right) + \frac{\varepsilon^2}{\sqrt{1+t}} \Bigg) \|\alpha_2 w_3^{\frac{1}{2}} \partial_q Z^N h\|_{L^2} + s.t.
\end{aligned}$$

We note that

$$\frac{d}{dt}\left(\frac{1}{\varepsilon(1+t)}\|\alpha_2 w_3^{\frac{1}{2}} \partial Z^N h\|_{L^2}^2 \right) \leq \frac{1}{\varepsilon(1+t)} \frac{d}{dt}\|\alpha_2 w_3^{\frac{1}{2}} \partial Z^N h\|_{L^2}^2$$

and we calculate

$$\frac{\varepsilon}{\varepsilon(1+t)^2}\|\alpha_2 w_3^{\frac{1}{2}} \partial_q Z^N h\|_{L^2}^2 \leq \frac{\varepsilon}{1+t} \frac{\|\alpha_2 w_3^{\frac{1}{2}} \partial_q Z^J h\|_{L^2}^2}{\varepsilon(1+t)},$$

$$\frac{\varepsilon}{\varepsilon(1+t)}\|\alpha_2 w_2'^{\frac{1}{2}}(q) \bar{\partial} Z^N \widetilde{g}_4\|_{L^2} \|\alpha_2 w_3^{\frac{1}{2}} \partial_q Z^N h\|_{L^2} \leq \sqrt{\varepsilon}\|\alpha_2 w_2'^{\frac{1}{2}}(q) \bar{\partial} \widetilde{Z}^I g_4\|_{L^2}^2 + \frac{1}{\sqrt{\varepsilon}(1+t)^2}\|\alpha_2 w_3^{\frac{1}{2}} \partial_q Z^N h\|_{L^2}^2,$$

$$\frac{\varepsilon}{\varepsilon(1+t)^{\frac{3}{2}}}\|\alpha_2 w_0^{\frac{1}{2}} \partial_q Z^J \varphi\|_{L^2} \|\alpha_2 w_3^{\frac{1}{2}} \partial_q Z^N h\|_{L^2} \leq \frac{\sqrt{\varepsilon}}{1+t}\|\alpha_2 w^{\frac{1}{2}} \partial_q Z^J \varphi\|_{L^2}^2 + \frac{1}{\sqrt{\varepsilon}(1+t)^2}\|\alpha_2 w_3^{\frac{1}{2}} \partial_q Z^N h\|_{L^2}^2.$$

This yields

$$
\begin{aligned}
&\frac{d}{dt}\left(\frac{1}{\varepsilon(1+t)}\|\alpha_2 w_3(q)^{\frac12}\partial\widetilde{Z^N h}\|^2_{L^2}\right)+\frac{1}{\varepsilon(1+t)}\|\alpha_2 w_3'(q)^{\frac12}\bar\partial\widetilde{Z^N h}\|^2_{L^2}\\
&\lesssim \frac{\sqrt{\varepsilon}}{1+t}E_N+\sqrt{\varepsilon}\|\alpha_2 w_2'^{\frac12}(q)\bar\partial\widetilde{Z}^I g_4\|^2_{L^2}+\frac{\varepsilon^{\frac52}}{1+t}.
\end{aligned}
\tag{3.10.22}
$$

The estimate for $Z^N k$ is totally similar. This, with the estimate (3.10.17) concludes the proof of Lemma 3.10.5. □

Proof of Lemma 3.10.6. We now go to the estimate for $Z^N\widetilde{g}_4$. We write $\Box_g Z^N\widetilde{g}_4=f_{\mu\nu}$. The energy estimate writes

$$
\begin{aligned}
\frac{d}{dt}\left(\|\alpha_2 w_2(q)^{\frac12}\partial Z^N\widetilde{g}_4\|^2_{L^2}\right)+\|\alpha_2 w_2'(q)^{\frac12}\bar\partial Z^N\widetilde{g}_4\|^2_{L^2}\lesssim&\|\alpha_2 w_2(q)^{\frac12}f_{\mu\nu}\|_{L^2}\|\alpha_2 w_2(q)^{\frac12}\partial Z^N\widetilde{g}_4\|_{L^2}\\
&+\frac{\varepsilon}{(1+t)}\|\alpha_2 w_2(q)^{\frac12}\partial Z^N\widetilde{g}_4\|^2_{L^2}
\end{aligned}
$$

We recall that the terms in $f_{\mu\nu}$ consist of

- the quasilinear terms,
- the terms coming from the non commutation of the wave operator with the null decomposition: it will be sufficient to study the term $\Upsilon(\frac{r}{t})\frac{1}{r^2}\partial_\theta Z^N h$,
- the semi-linear terms: it is sufficient to study the term $Z^N(g^{L\underline{L}}\partial_L g_{LL}\partial_{\underline{L}}h)$. We note that thanks to our decomposition, the term $Z^N(\partial_U g_{LL}\partial_{\underline{L}}h)$ is absent,
- The crossed terms: their analysis is similar to the one for φ.

The quasilinear terms We consider

$$
\left|\sum_{\substack{I+J\leq N\\ J\leq N-1}} Z^I g_{LL}\partial_q^2 Z^J\widetilde{g}_4\right|\lesssim\frac{1}{(1+|q|)}\sum_{\substack{I+J\leq N\\ J\leq N-1}} Z^I g_{LL}\partial_q Z^{J+1}\widetilde{g}_4|.
$$

If $I\leq\frac{N}{2}$, we can estimate

$$
|Z^I g_{LL}|\lesssim\frac{\varepsilon(1+|q|)}{(1+t)^{\frac32-\rho}},
$$

so

$$
\left\|\frac{\alpha_2 w_2^{\frac12}}{(1+|q|)}Z^I g_{LL}\partial_q Z^J\widetilde{g}_4\right\|_{L^2}\lesssim\frac{\varepsilon}{(1+t)^{\frac32-\rho}}\|\alpha_2 w_2^{\frac12}\partial_q Z^J\widetilde{g}_4\|_{L^2}.
\tag{3.10.23}
$$

If $J\leq\frac{N}{2}$, we can estimate, thanks to Proposition 3.4.8 and since the difference between $\widetilde{g}_4$ and $\widetilde{g}_3$ is contained in $\widetilde{g}_{\underline{L}U}$, which is equal to $(\widetilde{g}_3)_{\underline{L}U}$,

$$
\partial_q Z^J\widetilde{g}_4|\lesssim\frac{\varepsilon}{\sqrt{1+|q|}\sqrt{1+t}}.
\tag{3.10.24}
$$

Therefore, if we apply Hardy inequality we obtain

$$
\left\|\frac{\alpha_2 w_2^{\frac12}}{(1+|q|)}Z^I g_{LL}\partial_q Z^J\widetilde{g}_4\right\|_{L^2}\lesssim\frac{\varepsilon}{\sqrt{1+t}}\left\|\frac{\alpha_2 w_2^{\frac12}}{(1+|q|)^{\frac12}}\partial_q Z^I g_{LL}\right\|_{L^2}.
$$

Thanks to (3.10.4) and the fact that $\frac{w_2^{\frac{1}{2}}}{(1+|q|)^{\frac{1}{2}}} \lesssim w_2'(q)^{\frac{1}{2}}$ we obtain

$$\left\| \frac{\alpha_2 w_2^{\frac{1}{2}}}{(1+|q|)} Z^I g_{LL} \partial_q Z^J \widetilde{g}_4 \right\|_{L^2} \|\alpha_2 w_2(q)^{\frac{1}{2}} \partial Z^N \widetilde{g}_4\|_{L^2} \lesssim \frac{\varepsilon}{1+t} \|w_2(q)^{\frac{1}{2}} \partial Z^N \widetilde{g}_4\|^2_{L^2} + \varepsilon \|\alpha_2 w_2'^{\frac{1}{2}} \bar{\partial} \widetilde{g}_4\|^2_{L^2}. \tag{3.10.25}$$

The term coming from the non commutation of the wave operator with the null decomposition We note that $\frac{\partial_\theta h}{r}$ is a tangential derivative $\bar{\partial} h$. Therefore

$$\left\| \alpha_2 w_2^{\frac{1}{2}}(q) \Upsilon\left(\frac{r}{t}\right) \frac{1}{r^2} \partial_\theta Z^N h \right\|_{L^2} \lesssim \frac{1}{1+t} \|\alpha_2 w_2^{\frac{1}{2}} \bar{\partial} Z^N h\|_{L^2}$$

We calculate

$$\begin{cases} w_3'(q) = 2\mu \frac{1}{(1+|q|)^{1+2\mu}} \; for \; q<0, \\ w_3'(q) = (3+2\delta)(1+|q|)^{2+2\delta} \; for \; q>0. \end{cases}$$

Therefore $w_2 \lesssim w_3'$ and we obtain

$$\left\| w_2^{\frac{1}{2}}(q) \Upsilon\left(\frac{r}{t}\right) \frac{1}{r^2} \partial_\theta Z^N h \right\|_{L^2} \lesssim \frac{1}{1+t} \|\alpha_2 w_3'(q)^{\frac{1}{2}} \bar{\partial} Z^N h\|_{L^2}.$$

This yields

$$\begin{aligned} &\left\| \alpha_2 w_2^{\frac{1}{2}}(q) \Upsilon\left(\frac{r}{t}\right) \frac{1}{r^2} \partial_\theta Z^N h \right\|_{L^2} \|\alpha_2 w_2(q) \partial Z^N \widetilde{g}_4\|_{L^2} \\ &\lesssim \frac{1}{\sqrt{\varepsilon}(1+t)} \|\alpha_2 w_3'(q)^{\frac{1}{2}} \bar{\partial} Z^N h\|^2_{L^2} + \frac{\sqrt{\varepsilon}}{1+t} \|\alpha_2 w_2(q)^{\frac{1}{2}} \partial Z^N \widetilde{g}_4\|^2_{L^2}. \end{aligned} \tag{3.10.26}$$

The semi-linear terms We now estimate $Z^N(g^{L\underline{L}} \partial_L g_{LL} \partial_{\underline{L}} h)$. We first estimate

$$\|w_2(q)^{\frac{1}{2}} \bar{\partial} Z^{I_1} g_{LL} \partial Z^{I_2} h\|_{L^2}$$

for $I_1 + I_2 \leq N$ and $I_1 \leq N-1$. If $I_1 \leq \frac{N}{2}$ we estimate

$$|\bar{\partial} Z^{I_1} g_{LL}| \lesssim \frac{1}{1+s} |Z^{I_1+1} g_{LL}| \lesssim \frac{\varepsilon(1+|q|)}{(1+s)^{\frac{5}{2}-\rho}} \lesssim \frac{(1+|q|)^\rho}{(1+t)^{\frac{3}{2}}}.$$

Therefore

$$\|\alpha_2 w_2(q)^{\frac{1}{2}} \bar{\partial} Z^{I_1} g_{LL} \partial Z^{I_2} h\|_{L^2} \lesssim \frac{\varepsilon}{(1+t)^{\frac{3}{2}}} \|\alpha_2 w_2(q)^{\frac{1}{2}} (1+|q|)^\rho \partial Z^{I_2} h\|_{L^2} \lesssim \frac{\varepsilon}{(1+t)^{\frac{3}{2}}} \|\alpha_2 w_3(q)^{\frac{1}{2}} \partial Z^{I_2} h\|_{L^2}.$$

If $I_2 \leq \frac{N}{2}$ we estimate, thanks to (3.4.42)

$$|\partial Z^{I_2} h| \leq \frac{\varepsilon}{(1+|q|)^{\frac{3}{2}-\rho}},$$

therefore

$$\begin{aligned} \|\alpha_2 w_2(q)^{\frac{1}{2}} \bar{\partial} Z^{I_1} g_{LL} \partial Z^{I_2} h\|_{L^2} &\lesssim \varepsilon \left\| \frac{\alpha_2 w_2^{\frac{1}{2}}}{(1+|q|)^{\frac{3}{2}-\rho}} \bar{\partial} Z^{I_1} g_{LL} \right\|_{L^2} \\ &\lesssim \frac{\varepsilon}{1+t} \left\| \frac{\alpha_2 w_2^{\frac{1}{2}}}{(1+|q|)^{\frac{3}{2}-\rho}} Z^{I_1+1} g_{LL} \right\|_{L^2} \\ &\lesssim \frac{\varepsilon}{1+t} \left\| \frac{\alpha_2 w_2^{\frac{1}{2}}}{(1+|q|)^{\frac{1}{2}-\rho}} \partial Z^{I_1+1} g_{LL} \right\|_{L^2} \end{aligned}$$

where in the third inequality we have used the weighted Hardy inequality. Consequently

$$\|\alpha_2 w_2(q)^{\frac{1}{2}}\bar{\partial}Z^{I_1}g_{LL}\partial Z^{I_2}h\|_{L^2} \lesssim \frac{\varepsilon}{1+t}\|\alpha_2 w_2(q)^{\frac{1}{2}}\partial Z^{I_1+1}\widetilde{g}_4\|_{L^2}. \tag{3.10.27}$$

It is not possible to do the same reasoning for $I_1 = N$. To treat the term $g^{L\underline{L}}\partial_L Z^N g_{LL}\partial_{\underline{L}}h$, which appears only in $P_{\underline{L}L}$ we will write

$$\Box_g(hZ^N g_{LL}) = D^\alpha D_\alpha(hZ^N g_{LL}) = h\Box_g Z^N g_{LL} + Z^N(g_{LL})\Box_g h + g^{\alpha\beta}\partial_\alpha h\partial_\beta Z^N g_{LL}.$$

We estimate

$$\|w_2(q)^{\frac{1}{2}}\partial(hZ^N g_{LL})\|_{L^2} \lesssim \varepsilon\|w_2(q)^{\frac{1}{2}}\partial Z^N g_{LL}\|_{L^2}, \tag{3.10.28}$$

therefore, we can perform the energy estimate for $\widetilde{Z^N\widetilde{g}_4} = Z^N\widetilde{g}_4 - hZ^N g_{LL} - \frac{\chi(q)q\partial\widetilde{g}_4\partial_\theta^{N-1}b}{g_{UU}}$ instead of $Z^N\widetilde{g}_4$, where the last term is here to deal with the troublesome crossed term which is the equivalent of (3.10.7). We have now to estimate $h\Box_g Z^N g_{LL} + Z^N(g_{LL})\Box_g h + \partial Z^N g_{LL}\bar{\partial}h$. We estimate first

$$\|\alpha_2 w_2(q)^{\frac{1}{2}}\partial Z^N g_{LL}\bar{\partial}h\|_{L^2} \lesssim \frac{\varepsilon}{1+t}\|w_2(q)^{\frac{1}{2}}\partial Z^N g_{LL}\|_{L^2}. \tag{3.10.29}$$

We have $\Box_g h = -2(\partial_q\varphi)^2 + \partial_q h\partial_q g_{LL} + ...$ therefore

$$|\Box_g h| \lesssim \frac{\varepsilon^2}{(1+t)(1+|q|)}$$

and

$$\|\alpha_2 w_2(q)^{\frac{1}{2}}Z^N g_{LL}\Box_g h\|_{L^2} \lesssim \frac{\varepsilon}{1+t}\left\|\frac{\alpha_2 w_2(q)^{\frac{1}{2}}}{(1+|q|)}Z^N g_{LL}\right\| \lesssim \frac{\varepsilon}{1+t}\|w_2(q)^{\frac{1}{2}}\partial Z^N\widetilde{g}_4\|_{L^2}. \tag{3.10.30}$$

To estimate the last term, we have to note that since $g^{L\underline{L}}\partial_L Z^N g_{LL}\partial_{\underline{L}}h$ appears only in $P_{L\underline{L}}$, it is absent from $\Box_g Z^N g_{LL}$. However, we have terms appearing from the non commutation of the wave operator with the null decomposition. They are of the form $\frac{1}{r}h\bar{\partial}Z^N g_{LL}$. We estimate

$$\left\|\alpha_2 w_2(q)^{\frac{1}{2}}\frac{1}{r}h\bar{\partial}Z^N g_{LL}\right\|_{L^2} \lesssim \frac{\varepsilon}{1+t}\|\alpha_2 w_2(q)^{\frac{1}{2}}\partial Z^N\widetilde{g}_4\|_{L^2} \tag{3.10.31}$$

The other terms in $\Box_g Z^N g_{LL}$ have already been estimated.

Remark 3.10.8. *This reasoning would not have been possible to treat terms of the form $\partial_U g_{LL}\partial_q h$. It is why we have introduced the function k, which is allowed to decay less.*

Thanks to (3.10.23), (3.10.25), (3.10.26), (3.10.27), (3.10.29), (3.10.30), (3.10.31) the energy estimate yields

$$\begin{aligned}
&\frac{d}{dt}\left(\|\alpha_2 w_2(q)^{\frac{1}{2}}\partial\widetilde{Z^N\widetilde{g}_4}\|_{L^2}^2\right) + \|\alpha_2 w_2'(q)^{\frac{1}{2}}\bar{\partial}\widetilde{Z^N\widetilde{g}_4}\|_{L^2}^2 \\
\lesssim&\frac{\sqrt{\varepsilon}}{1+t}\left(\|\alpha_2 w_2(q)^{\frac{1}{2}}\partial Z^N\widetilde{g}_4\|_{L^2}^2 + \frac{1}{\varepsilon(1+t)}\|\alpha_2 w_3^{\frac{1}{2}}\partial h\|_{L^2}^2\right) \\
&+\sqrt{\varepsilon}\left(\frac{1}{\varepsilon(1+t)}\|\alpha_2 w_3'(q)^{\frac{1}{2}}\bar{\partial}Z^N h\|_{L^2}^2 + \|\alpha_2 w_2'(q)^{\frac{1}{2}}\bar{\partial}Z^N\widetilde{g}_4\|_{L^2}^2\right) + s.t.
\end{aligned} \tag{3.10.32}$$

This, together with the estimates (3.10.28) concludes the proof of Lemma 3.10.6. $\Box$

3.10.2 Estimates for $I \leq N-2$

Proposition 3.10.9. *Let $I \leq N-2$. We have the estimates*

$$\begin{aligned}
\|\alpha w_0(q)^{\frac{1}{2}}\partial Z^I\varphi\|_{L^2} &\leq C_0\varepsilon + C\varepsilon^{\frac{3}{2}},\\
\|\alpha w(q)^{\frac{1}{2}}\partial Z^I h\|_{L^2} &\leq C\varepsilon^{\frac{3}{2}}(1+t),\\
\|\alpha w_2(q)^{\frac{1}{2}}\partial Z^I\widetilde{g}_3\|_{L^2} &\leq C_0\varepsilon + C\varepsilon^{\frac{5}{4}}.
\end{aligned}$$

Moreover

$$\int_0^t \|\alpha w_2'(q)^{\frac{1}{2}}\bar{\partial}Z^I\widetilde{g}_3\|_{L^2}^2 \lesssim \varepsilon^2.$$

We prove the proposition by using the energy estimate for φ, h and $\widetilde{g}_3$.

Proposition 3.10.10. *Let $I \leq N-2$. We have*

$$\frac{d}{dt}\sum_{J\leq I}\|\alpha w_0(q)^{\frac{1}{2}}\partial Z^J\varphi\|_{L^2}^2+\sum_{J\leq I}\|\alpha w_0'(q)^{\frac{1}{2}}\bar{\partial}Z^J\varphi\|_{L^2}^2 \lesssim \varepsilon^3\frac{1}{(1+t)^{1+\sigma-C\sqrt{\varepsilon}}}+\frac{\varepsilon}{(1+t)^{\sigma}}\sum_{J\leq I}\left\|w_0'(q)^{\frac{1}{2}}\bar{\partial}Z^J\varphi\right\|_{L^2}^2.$$

Proposition 3.10.11. *Let $I \leq N-2$. We have the estimate*

$$\frac{d}{dt}\sum_{J\leq I}\|\alpha w_3(q)^{\frac{1}{2}}\partial Z^J h\|_{L^2}^2 + \sum_{J\leq I}\|\alpha w_3'(q)\bar{\partial}Z^J h\|_{L^2}^2 \lesssim \varepsilon^3.$$

Proposition 3.10.12. *Let $I \leq N-2$. We have*

$$\frac{d}{dt}\sum_{J\leq I}\|\alpha w_2(q)^{\frac{1}{2}}\partial Z^J\widetilde{g}_3\|_{L^2}^2 + \sum_{J\leq I}\|\alpha w_2'(q)^{\frac{1}{2}}\bar{\partial}Z^J\widetilde{g}_3\|_{L^2}^2 \lesssim \frac{\varepsilon^{\frac{5}{2}}}{(1+t)^{1+\sigma}} + \frac{\varepsilon}{(1+t)^{\sigma}}\|w_2'(q)^{\frac{1}{2}}\bar{\partial}Z^{I+1}\widetilde{g}_3\|_{L^2}^2.$$

We admit for the moment Propositions 3.10.10, 3.10.11 and 3.10.12 and prove Proposition 3.10.9.

Proof of Proposition 3.10.9. We estimate φ. Since $\sigma > C\sqrt{\varepsilon}$ for $\varepsilon > 0$ small enough, by integrating the inequality of Proposition 3.10.10 with respect to t we obtain

$$\begin{aligned}
&\sum_{J\leq I}\|\alpha w_0(q)^{\frac{1}{2}}\partial Z^J\varphi\|_{L^2}^2 + \int_0^t\sum_{J\leq I}\|\alpha w_0'(q)^{\frac{1}{2}}\bar{\partial}Z^J\varphi\|_{L^2}^2\\
&\leq \sum_{J\leq I}\|\alpha w_0(q)^{\frac{1}{2}}\partial Z^I\varphi(0)\|_{L^2}^2 + C\varepsilon^3 + C\int_0^t\frac{\varepsilon}{(1+\tau)^{\sigma}}\sum_{J\leq I}\left\|w_0'(q)^{\frac{1}{2}}\bar{\partial}Z^J\varphi\right\|_{L^2}^2.
\end{aligned}$$

Thanks to Proposition 3.10.2, we have

$$\int_0^t\frac{\varepsilon}{(1+\tau)^{\sigma}}\sum_{J\leq I}\left\|w_0'(q)^{\frac{1}{2}}\bar{\partial}Z^J\varphi\right\|_{L^2}^2 \lesssim \varepsilon^2,$$

and therefore

$$\sum_{J\leq I}\|\alpha w_0(q)^{\frac{1}{2}}\partial Z^J\varphi\|_{L^2}^2 \lesssim C_0^2\varepsilon^2 + C\varepsilon^3.$$

We now estimate h. We integrate the inequality of Proposition 3.10.11 with respect to t. We obtain, since we take zero initial data for h, and therefore, initial data for $Z^I h$ of size ε^2

$$\sum_{J\leq I}\|\alpha w_3(q)^{\frac{1}{2}}\partial Z^J h\|_{L^2}^2 + \int_0^t\sum_{J\leq I}\|\alpha w_3'(q)\bar{\partial}Z^J h\|_{L^2}^2 \lesssim \varepsilon^3(1+t).$$

We now integrate the inequality of Proposition 3.10.12 to estimate $\widetilde{g}_3$. We obtain

$$\begin{aligned}
&\sum_{J\leq I}\|\alpha w_2(q)^{\frac12}\partial Z^J\widetilde{g}_3\|_{L^2}^2+\int_0^t\sum_{J\leq I}\|\alpha w_2'(q)^{\frac12}\bar{\partial}Z^J\widetilde{g}_3\|_{L^2}^2\\
&\leq\sum_{J\leq I}\|\alpha w_2(q)^{\frac12}\partial Z^J\widetilde{g}_3(0)\|_{L^2}^2+C\varepsilon^{\frac52}+\int_0^t\frac{\varepsilon}{(1+\tau)^{\sigma}}\|\alpha w_2'(q)^{\frac12}\bar{\partial}Z^{I+1}\widetilde{g}_4\|_{L^2}^2.
\end{aligned}$$

Proposition 3.10.2 yields

$$\int_0^t\frac{1}{(1+t)^{\sigma}}\|w_2'(q)^{\frac12}\bar{\partial}Z^{I+1}\widetilde{g}_4\|_{L^2}^2\lesssim\varepsilon^2,$$

Therefore

$$\sum_{I\leq N-1}\|\alpha w_2(q)^{\frac12}\partial Z^I\widetilde{g}_3\|_{L^2}^2+\int_0^t\sum_{I\leq N-1}\|\alpha w_2'(q)^{\frac12}\bar{\partial}Z^I\widetilde{g}_3\|_{L^2}^2\leq C_0^2\varepsilon^2+C\varepsilon^{\frac52}.$$

This concludes the proof of Proposition 3.10.9. □

Proof of Proposition 3.10.10. We follow the proof of Lemma 3.10.4. Let $I\leq N-1$. We use the weighted energy estimate for the equation (3.10.2). It yields

$$\begin{aligned}
\frac{d}{dt}\left(\|\alpha w(q)^{\frac12}\partial Z^I\varphi\|_{L^2}^2\right)+\|\alpha w'(q)^{\frac12}\bar{\partial}Z^I\varphi\|_{L^2}^2&\lesssim\left\|\Box_g Z^I\varphi\right\|_{L^2}\|\alpha w(q)^{\frac12}\partial Z^I\varphi\|_{L^2}\\
&\quad+\frac{\varepsilon}{(1+t)^{1+\sigma}}\|w^{\frac12}\partial Z^I\varphi\|_{L^2}^2.
\end{aligned}\tag{3.10.33}$$

We first estimate

$$\left|\sum_{\substack{I_1+I_2\leq I\\ I_2\leq I-1}}Z^{I_1}g_{LL}\partial_q^2Z^{I_2}\varphi\right|\lesssim\frac{1}{(1+|q|)}\sum_{I_1+I_2\leq I}|Z^{I_1}g_{LL}\partial_qZ^{I_2}\varphi|.$$

If $I_1\leq\frac{N}{2}$, we can estimate

$$|Z^{I_1}g_{LL}|\lesssim\frac{\varepsilon(1+|q|)}{(1+t)^{\frac32-\rho}}$$

so

$$\left\|\frac{\alpha w_0^{\frac12}}{(1+|q|)}Z^{I_1}g_{LL}\partial_qZ^{I_2}\varphi\right\|_{L^2}\lesssim\frac{\varepsilon}{(1+t)^{\frac32-\rho}}\|\alpha w_0(q)^{\frac12}\partial_qZ^{I_2}\varphi\|_{L^2}.\tag{3.10.34}$$

If $I_2\leq\frac{N}{2}$, we can estimate

$$|\partial_qZ^{I_2}\varphi|\lesssim\frac{\varepsilon}{(1+|q|)^{\frac32-4\rho}\sqrt{1+t}},\ for\ q<0,\quad|\partial_qZ^{I_2}\varphi|\lesssim\frac{\varepsilon}{(1+|q|)^{\frac32+\delta-\sigma}\sqrt{1+t}},\ for\ q>0.$$

We apply the weighted Hardy inequality, but in order to be in its range, we cannot keep all the decay in q in the region $q>0$.

$$\left\|\frac{\alpha w_0(q)^{\frac12}}{(1+|q|)}Z^{I_1}g_{LL}\partial_qZ^{I_2}\varphi\right\|_{L^2}\lesssim\frac{\varepsilon}{\sqrt{1+t}}\left\|\frac{v(q)^{\frac12}}{(1+|q|)}Z^Ig_{LL}\right\|_{L^2}\lesssim\frac{\varepsilon}{\sqrt{1+t}}\|v(q)^{\frac12}\partial_qZ^{I_1}g_{LL}\|_{L^2}$$

where

$$\begin{cases}v(q)=\frac{1}{(1+|q|)^{3-4\rho}}\ for\ q<0,\\ v(q)=\frac{w_0(q)}{(1+|q|)^{1+\delta}}=(1+|q|)^{1+\delta}\ for\ q>0.\end{cases}\tag{3.10.35}$$

We use (3.10.4), which gives $\partial_q Z^{I_1} g_{LL} \sim \bar{\partial} Z^{I_1} \widetilde{g}_4$ so

$$|\partial_q Z^{I_1} g_{LL}| \lesssim \frac{1}{1+s}|Z^{I_1+1}\widetilde{g}_4| \lesssim \frac{1}{(1+t)^{\frac{1}{2}+\sigma}(1+|q|)^{\frac{1}{2}-\sigma}}|Z^{I_1+1}\widetilde{g}_4|. \tag{3.10.36}$$

Therefore, we obtain

$$\begin{aligned}\left\|\frac{\alpha w_0(q)^{\frac{1}{2}}}{(1+|q|)} Z^{I_1} g_{LL} \partial_q Z^{I_2}\varphi\right\|_{L^2} &\lesssim \frac{\varepsilon}{(1+t)^{1+\sigma}}\left\|\frac{v(q)^{\frac{1}{2}}}{(1+|q|)^{\frac{1}{2}-\sigma}} Z^{I_1+1}\widetilde{g}_4\right\|_{L^2} \\ &\lesssim \frac{\varepsilon}{(1+t)^{1+\sigma}}\|v^{\frac{1}{2}}(1+|q|)^{\frac{1}{2}+\sigma}\partial_q Z^{I_1+1}\widetilde{g}_4\|_{L^2}\end{aligned}$$

where we have used again the weighted Hardy inequality. We calculate

$$\begin{cases} v(q)(1+|q|)^{1+2\sigma} = \frac{1}{(1+|q|)^{2-4\rho-2\sigma}} \; for \; q<0, \\ v(q)(1+|q|)^{1+2\sigma} = (1+|q|)^{2+\delta+2\sigma} \; for \; q>0. \end{cases}$$

Therefore if $1-4\rho-2\sigma \geq \mu$ and $\delta + 2\sigma < 2\delta$ we have $v(q)(1+|q|)^{1+2\sigma} \leq w_2$ so we obtain, together with Proposition 3.10.2,

$$\left\|\frac{\alpha w_0(q)^{\frac{1}{2}}}{(1+|q|)} Z^{I_1} g_{LL} \partial_q Z^{I_2}\varphi\right\|_{L^2} \lesssim \frac{\varepsilon}{(1+t)^{1+\sigma}}\|w_2(q)^{\frac{1}{2}}\partial_q Z^{I_1+1}\widetilde{g}_4\|_{L^2} \lesssim \frac{\varepsilon^2(1+t)^{C\sqrt{\varepsilon}}}{(1+t)^{1+\sigma}} \tag{3.10.37}$$

We now estimate the crossed terms, for which the weight modulator α has been introduced. They are of the form (3.2.13). It is sufficient to estimate, for $I \leq N-1$

$$\left\|\frac{\varepsilon \mathbb{1}_{q>0}}{r}\alpha(q) w_0(q)^{\frac{1}{2}}\bar{\partial} Z^I \varphi\right\|_{L^2}.$$

We obtain

$$\left\|\frac{\varepsilon \mathbb{1}_{q>0}}{r}\alpha(q) w_0(q)^{\frac{1}{2}}\bar{\partial} Z^{N-1} \varphi\right\|_{L^2} \lesssim \frac{\varepsilon}{(1+t)^{\frac{1}{2}+\sigma}}\left\|\frac{\mathbb{1}_{q>0}}{(1+|q|)^{\frac{1}{2}}} w_0(q)^{\frac{1}{2}}\bar{\partial} Z^{N-1} \varphi\right\|_{L^2}.$$

and consequently, since in the region $q>0$ we have $\frac{w_0(q)^{\frac{1}{2}}}{\sqrt{1+|q|}} \lesssim w_0'(q)^{\frac{1}{2}}$,

$$\left\|\frac{\varepsilon \mathbb{1}_{q>0}}{r}\alpha(q) w_0(q)^{\frac{1}{2}}\bar{\partial} Z^{N-1} \varphi\right\|_{L^2}\|\alpha w_0^{\frac{1}{2}}\partial Z^I\varphi\|_{L^2}^2 \lesssim \frac{\varepsilon}{(1+t)^{1+\sigma}}\|\alpha w_0^{\frac{1}{2}}\partial Z^I\varphi\|^2_{L^2} + \frac{\varepsilon}{(1+t)^{\sigma}}\|w_0'(q)^{\frac{1}{2}}\bar{\partial} Z^{N-1}\varphi\|^2_{L^2} \tag{3.10.38}$$

The last term which appears in (3.10.33) can be estimated thanks to Proposition 3.10.2

$$\frac{\varepsilon}{(1+t)^{1+\sigma}}\|w_0^{\frac{1}{2}}\partial Z^I\varphi\|^2_{L^2} \lesssim \frac{\varepsilon^3}{(1+t)^{1+\sigma-C\sqrt{\varepsilon}}}. \tag{3.10.39}$$

The estimates (3.10.33), (3.10.34), (3.10.37), (3.10.38) and (3.10.39), together with the bootstrap assumption (3.4.25) which imply

$$\|\alpha w_0(q)^{\frac{1}{2}}\partial Z^I\varphi\|_{L^2} \lesssim \varepsilon,$$

conclude the proof of Proposition 3.10.10. ☐

We now estimate h

Proof of 3.10.11. The equation for $Z^I h$ is given by (3.10.12). We estimate first

$$\left\| \alpha w_3^{\frac{1}{2}} \sum_{\substack{I_1+I_2\leq I \\ I_2\leq I-1}} \left(Z^{I_1} g^{\alpha\beta}\right)\left(Z^{I_2}\partial_\alpha\partial_\beta h\right)\right\|_{L^2}.$$

As before, we can estimate, for $I_1 \leq \frac{N}{2}$, thanks to the bootstrap assumption (3.4.25),

$$\left\| \frac{\alpha w_3^{\frac{1}{2}}}{(1+|q|)} Z^{I_1} g_{LL}\partial_q Z^{I_2} h\right\|_{L^2} \lesssim \frac{\varepsilon}{(1+t)^{\frac{3}{2}-\rho}} \|\alpha w_3^{\frac{1}{2}}\partial_q Z^{I_2} h\|_{L^2} \lesssim \frac{\varepsilon^2}{(1+t)^{1-\rho}}. \tag{3.10.40}$$

For $I_2 \leq \frac{N}{2}$, we have the estimate

$$|\partial_q Z^{I_2} h| \lesssim \frac{\varepsilon}{(1+|q|)^{\frac{3}{2}-\rho}},\ for\ q<0, \quad |\partial_q Z^{I_2} h| \lesssim \frac{\varepsilon}{(1+|q|)^{2+\delta-\sigma}},\ for\ q<0,$$

so

$$\left\| \frac{\alpha w_3^{\frac{1}{2}}}{(1+|q|)} Z^{I_1} g_{LL}\partial_q Z^{I_2} h\right\|_{L^2} \lesssim \left\| \frac{\alpha v^{\frac{1}{2}}}{(1+|q|)} Z^{I_1} g_{LL}\right\|_{L^2}.$$

where v is defined by (3.10.35) and with Hardy inequality and the same reasoning than for φ

$$\left\| \frac{\alpha w_3^{\frac{1}{2}}}{(1+|q|)} Z^{I_1} g_{LL}\partial_q Z^{I_2} h\right\|_{L^2} \lesssim \varepsilon \|v^{\frac{1}{2}}\partial_q Z^{I_1} g_{LL}\|_{L^2} \lesssim \varepsilon \frac{1}{(1+t)^{\frac{1}{2}+\sigma}} \|w_2(q)^{\frac{1}{2}}\partial Z^{I_1+1}\widetilde{g}_4\|_{L^2},$$

and thanks to Proposition 3.10.2 we obtain

$$\left\| \frac{\alpha w_3^{\frac{1}{2}}}{(1+|q|)} Z^{I_1} g_{LL}\partial_q Z^{I_2} h\right\|_{L^2} \lesssim \frac{\varepsilon^2(1+t)^{C\sqrt{\varepsilon}}}{(1+t)^{\frac{1}{2}+\sigma}}. \tag{3.10.41}$$

We estimate the second term

$$\left\|\alpha w_3^{\frac{1}{2}} Z^I(\partial_q\varphi)^2\right\|_{L^2} \lesssim \sum_{I_1+I_2\leq I} \|\alpha w_3^{\frac{1}{2}}\partial_q Z^{I_1}\varphi\partial_q Z^{I_2}\varphi\|_{L^2} \lesssim \frac{\varepsilon}{\sqrt{1+t}} \sum_{J\leq I}\|\alpha w_0^{\frac{1}{2}}\partial_q Z^I\varphi\|_{L^2}$$

so thanks to (3.4.25) we obtain

$$\left\|\alpha w_3^{\frac{1}{2}} Z^I(\partial_q\varphi)^2\right\|_{L^2} \lesssim \frac{\varepsilon^2}{\sqrt{1+t}}. \tag{3.10.42}$$

The semi-linear term $\partial_{\underline{L}} g_{LL}\partial_{\underline{L}} h$, appearing in $Q_{\underline{L}\underline{L}}$ can be estimated in the same way at the first. The crossed term $\partial_{\underline{L}}(g_b)_{UU}\partial_{\underline{L}} g_{\underline{L}\underline{L}}$ appearing in $Q_{\underline{L}\underline{L}}$ and the term $(R_b)_{qq}$ can be estimated in the same way than in the case $I\leq N$. The crossed terms of $H_b^\rho\partial_\rho h$ can be estimated in the following way

$$\left\|\frac{\varepsilon \mathbb{1}_{q>0}}{r}\alpha w_3^{\frac{1}{2}}\partial Z^I h\right\|_{L^2} \lesssim \frac{\varepsilon}{\sqrt{1+t}}\|\alpha w_3^{\frac{1}{2}}\partial Z^I h\| \lesssim \frac{\varepsilon^2}{\sqrt{1+t}}. \tag{3.10.43}$$

Thanks to (3.10.40), (3.10.41), (3.10.42) and (3.10.43), and the bootstrap assumption (3.4.25), the energy inequality yields (we use here the first inequality of Proposition 3.9.1)

$$\frac{d}{dt}\|\alpha w_3^{\frac{1}{2}}\partial Z^I h\|_{L^2}^2 + \|\alpha w_3'(q)^{\frac{1}{2}}\bar\partial Z^I h\|_{L^2}^2 \lesssim \frac{\varepsilon^2}{\sqrt{1+t}}\|\alpha w_3^{\frac{1}{2}}\partial_q Z^I h\|_{L^2} + \frac{\varepsilon}{1+t}\|\alpha w_3^{\frac{1}{2}}\partial_q Z^I h\|_{L^2}^2 \lesssim \varepsilon^3,$$

which concludes the proof of Proposition 3.10.11. $\square$

We now estimate $\widetilde{g}_3$

Proof of Proposition 3.10.12. We write $\Box_g Z^I \widetilde{g}_3 = f_{\mu\nu}$. The energy estimate yields

$$\frac{d}{dt}\left(\|\alpha w_2(q)^{\frac{1}{2}}\partial Z^I\widetilde{g}_3\|_{L^2}^2\right) + \|\alpha w'(q)^{\frac{1}{2}}\bar{\partial} Z^I\widetilde{g}_3\|_{L^2}^2 \lesssim \|\alpha w_2(q) f_{\mu\nu}\|_{L^2}\|\alpha w_2(q)\partial Z^I\widetilde{g}_3\|_{L^2} + \frac{\varepsilon}{(1+t)^\sigma}\|w_2(q)^{\frac{1}{2}}\partial Z^I\widetilde{g}_3\|_{L^2}^2.$$

We recall that the terms in $f_{\mu\nu}$ are

- the quasilinear terms,
- the terms coming from the non commutation of the wave operator with the null decomposition: it will be sufficient to study the term $\chi(\frac{r}{t})\frac{1}{r^2}\partial_\theta Z^N h$,
- the semi-linear terms: it is sufficient to study the term $Z^I\partial_U g_{LL}\partial_{\underline{L}} h$,
- the crossed term: their analysis is the same than for φ.

We first estimate the quasilinear term.

$$\left|\sum_{\substack{I_1+I_2\leq I\\ I_2\leq I-1}} Z^{I_1} g_{LL}\partial_q^2 Z^{I_2}\widetilde{g}_4\right| \lesssim \frac{1}{(1+|q|)}\sum_{I_1+I_2\leq I}|Z^{I_1}g_{LL}\partial_q Z^{I_2}\widetilde{g}_4|$$

If $I_1 \leq \frac{N}{2}$, we can estimate

$$|Z^{I_1}g_{LL}| \lesssim \frac{\varepsilon(1+|q|)}{(1+t)^{\frac{3}{2}-\rho}}$$

so

$$\left\|\frac{\alpha w_2^{\frac{1}{2}}}{(1+|q|)} Z^{I_1} g_{LL}\partial_q Z^{I_2}\widetilde{g}_4\right\|_{L^2} \lesssim \frac{\varepsilon}{(1+t)^{\frac{3}{2}-\rho}}\|\alpha w_2^{\frac{1}{2}}\partial_q Z^{I_2}\widetilde{g}_4\|_{L^2} \tag{3.10.44}$$

If $I_2 \leq \frac{N}{2}$, we can estimate, thanks (3.10.24) and (3.4.42)

$$|\partial_q Z^{I_2}\widetilde{g}_4| \lesssim \left(\frac{\varepsilon}{\sqrt{1+t}\sqrt{1+|q|}}\right)^{\frac{1}{2}}\left(\frac{\varepsilon}{(1+|q|)^{\frac{3}{2}-\rho}}\right)^{\frac{1}{2}} \lesssim \frac{\varepsilon}{(1+t)^{\frac{1}{4}}(1+|q|)^{1-\frac{\rho}{2}}}$$

Therefore,

$$\left\|\frac{\alpha w_2^{\frac{1}{2}}}{(1+|q|)} Z^{I_1} g_{LL}\partial_q Z^{I_2}\widetilde{g}_4\right\|_{L^2} \lesssim \frac{\varepsilon}{t^{\frac{1}{4}}}\left\|\frac{\alpha w_2^{\frac{1}{2}}}{(1+|q|)^{2-\frac{\rho}{2}}} Z^{I_1} g_{LL}\right\|_{L^2} \lesssim \frac{\varepsilon}{t^{\frac{1}{4}}}\left\|\frac{\alpha w_2^{\frac{1}{2}}}{(1+|q|)^{1-\frac{\rho}{2}}}\partial Z^{I_1} g_{LL}\right\|_{L^2},$$

where we have used the weighted Hardy inequality, noting that in the region $q>0$

$$\frac{\alpha^2 w_2}{(1+|q|)^{2-\rho}} = (1+|q|)^{2\delta-2\sigma-\rho},$$

so the condition $\delta > \sigma+\rho+\frac{1}{2}$ ensure that we can apply the weighted Hardy inequality. We use the wave coordinate condition, (3.10.4) which gives $\partial_q Z^{I_1} g_{LL} \sim \bar{\partial} Z^{I_1}\widetilde{g}_4$. We obtain

$$|\partial_q Z^{I_1} g_{LL}| \lesssim \frac{1}{1+s}|Z^{I_1+1}\widetilde{g}_4|\frac{1}{(1+t)^{\frac{3}{4}+\sigma}(1+|q|)^{\frac{1}{4}-\sigma}}|Z^{I_1+1}\widetilde{g}_4|.$$

It yields, by using Hardy inequality again

$$\left\| \frac{\alpha w_2^{\frac{1}{2}}}{(1+|q|)} Z^{I_1} g_{LL} \partial_q Z^{I_2} \widetilde{g}_4 \right\|_{L^2} \lesssim \frac{\varepsilon}{(1+t)^{1+\sigma}} \left\| \frac{\alpha w_2^{\frac{1}{2}}}{(1+|q|)^{\frac{5}{4}-\frac{\rho}{2}-\sigma}} Z^{I_1+1} \widetilde{g}_4 \right\|_{L^2}$$
$$\lesssim \frac{\varepsilon}{(1+t)^{1+\sigma}} \| \frac{\alpha w_2^{\frac{1}{2}}}{(1+|q|)^{\frac{1}{4}-\frac{\rho}{2}-\sigma}} Z^{I_1+1} \widetilde{g}_4 \|_{L^2}.$$

Consequently, since $\frac{1}{4} - \frac{\rho}{2} - \sigma > 0$ we have

$$\left\| \frac{\alpha w_2^{\frac{1}{2}}}{(1+|q|)} Z^{I_1} g_{LL} \partial_q Z^{I_2} \widetilde{g}_4 \right\|_{L^2} \lesssim \frac{\varepsilon}{(1+t)^{1+\sigma}} \| w_2^{\frac{1}{2}} \partial Z^{I+1} \widetilde{g}_4 \|_{L^2} \lesssim \frac{\varepsilon^2 (1+t)^{C\sqrt{\varepsilon}}}{(1+t)^{1+\sigma}}, \tag{3.10.45}$$

where we have used Proposition 3.10.2. We now estimate the term coming from the non commutation with the wave operator $\|\alpha w_2^{\frac{1}{2}}(q) \Upsilon\left(\frac{r}{t}\right) \frac{1}{r^2} \partial_\theta Z^I h\|_{L^2}$. On the support of $\Upsilon\left(\frac{r}{t}\right)$, we have $r \sim t$ and hence

$$\frac{1}{r^2} \lesssim \frac{1}{(1+t)^{\frac{3}{2}+\sigma}(1+|q|)^{\frac{1}{2}-\sigma}}.$$

Therefore

$$\left\| \alpha w_2^{\frac{1}{2}}(q) \Upsilon\left(\frac{r}{t}\right) \frac{1}{r^2} \partial_\theta Z^I h \right\|_{L^2} \lesssim \frac{1}{(1+t)^{\frac{3}{2}+\sigma}} \left\| \frac{\alpha w_2^{\frac{1}{2}}}{(1+|q|)^{\frac{1}{2}-\sigma}} Z^{I+1} h \right\|_{L^2}$$
$$\lesssim \frac{1}{(1+t)^{\frac{3}{2}+\sigma}} \left\| \alpha w_2^{\frac{1}{2}} (1+|q|)^{\frac{1}{2}+\sigma} \partial Z^{I+1} h \right\|_{L^2},$$

where we have applied the weighted Hardy inequality. We calculate

$$\alpha^2 w_2 (1+|q|)^{1+2\sigma} = \begin{cases} (1+|q|)^{2\sigma-2\mu} \; for \; q<0, \\ (1+|q|)^{3+2\delta} \; for \; q>0. \end{cases}$$

If $\sigma < \mu$ we have $\alpha^2 w_2 (1+|q|)^{1+2\sigma} \le w_3$ and

$$\left\| \alpha w_2^{\frac{1}{2}}(q) \chi\left(\frac{r}{t}\right) \frac{1}{r^2} \partial_\theta Z^I h \right\|_{L^2} \lesssim \frac{1}{(1+t)^{\frac{3}{2}+\sigma}} \| w_3(q)^{\frac{1}{2}} \partial Z^{I+1} h \|_{L^2} \lesssim \frac{\varepsilon^{\frac{3}{2}} (1+t)^{C\sqrt{\varepsilon}}}{(1+t)^{1+\sigma}}, \tag{3.10.46}$$

where we have used Proposition 3.10.2 which yields, for $I \le N-2$

$$\| w_3(q)^{\frac{1}{2}} \partial Z^{I+1} h \|_{L^2} \lesssim \varepsilon \frac{3}{2} (1+t)^{\frac{1}{2}+C\sqrt{\varepsilon}}.$$

We now estimate $Z^I(\partial_U g_{LL} \partial_{\underline{L}} h)$. We have

$$\| \alpha w_2(q)^{\frac{1}{2}} Z^I (\partial_U g_{LL} \partial_{\underline{L}} h) \|_{L^2} \lesssim \sum_{I_1+I_2 \le I} \| \alpha w_2(q)^{\frac{1}{2}} \bar{\partial} Z^{I_1} g_{LL} \partial Z^{I_2} h \|_{L^2}.$$

If $I_1 \le \frac{N}{2}$ we estimate

$$|\bar{\partial} Z^{I_1} g_{LL}| \lesssim \frac{1}{1+s} |Z^{I_1+1} g_{LL}| \lesssim \frac{\varepsilon(1+|q|)}{(1+t)^{\frac{5}{2}-\rho}} \lesssim \frac{(1+|q|)^{\rho+\sigma}}{(1+t)^{\frac{3}{2}+\sigma}}.$$

Therefore

$$\| \alpha w_2(q)^{\frac{1}{2}} \bar{\partial} Z^{I_1} g_{LL} \partial Z^{I_2} h \|_{L^2} \lesssim \frac{\varepsilon}{(1+t)^{\frac{3}{2}+\sigma}} \| \alpha w_2^{\frac{1}{2}} (1+|q|)^{\rho+\sigma} \partial Z^{I_2} h \|_{L^2} \lesssim \frac{\varepsilon}{t^{\frac{3}{2}+\sigma}} \| \alpha w_3(q)^{\frac{1}{2}} \partial Z^{I_2} h \|_{L^2},$$

and consequently

$$\|\alpha w_2(q)^{\frac{1}{2}}\bar{\partial}Z^{I_1}g_{LL}\partial Z^{I_2}h\|_{L^2} \lesssim\lesssim \frac{\varepsilon^2}{t^{1+\sigma}}. \tag{3.10.47}$$

If $I_2 \leq \frac{N}{2}$ thanks to (3.4.42) we estimate

$$|\partial Z^{I_2}h| \leq \frac{\varepsilon}{(1+|q|)^{\frac{3}{2}-\rho}},$$

therefore

$$\begin{aligned}
\|\alpha w_2(q)^{\frac{1}{2}}\bar{\partial}Z^{I_1}g_{LL}\partial Z^{I_2}h\|_{L^2} &\lesssim \varepsilon\left\|\frac{\alpha w_2^{\frac{1}{2}}}{(1+|q|)^{\frac{3}{2}-\rho}}\bar{\partial}Z^{I_1}g_{LL}\right\|_{L^2}\\
&\lesssim \frac{\varepsilon}{(1+t)^{\frac{1}{2}+\sigma}}\left\|\frac{\alpha w_2^{\frac{1}{2}}}{(1+|q|)^{2-\rho-\sigma}}Z^{I_1+1}g_{LL}\right\|_{L^2}\\
&\lesssim \frac{\varepsilon}{(1+t)^{\frac{1}{2}+\sigma}}\left\|\frac{\alpha w_2^{\frac{1}{2}}}{(1+|q|)^{1-\rho-\sigma}}\partial Z^{I_1+1}g_{LL}\right\|_{L^2}\\
&\lesssim \frac{\varepsilon}{(1+t)^{\frac{1}{2}+\sigma}}\|\alpha w_2'(q)^{\frac{1}{2}}\bar{\partial}Z^{I_1+1}\widetilde{g}_4\|_{L^2}
\end{aligned}$$

where in the last inequality we have used the wave coordinate condition. Therefore

$$\|\alpha w_2(q)^{\frac{1}{2}}\bar{\partial}Z^{I_1}g_{LL}\partial Z^{I_2}h\|_{L^2}\|\alpha w_2(q)^{\frac{1}{2}}\partial Z^{I}\widetilde{g}_3\| \lesssim \frac{\varepsilon}{(1+t)^{\sigma}}\|\alpha w_2'(q)^{\frac{1}{2}}\bar{\partial}Z^{I_1+1}\widetilde{g}_4\|_{L^2}^2 + \frac{\varepsilon^3}{(1+t)^{1+\sigma}}. \tag{3.10.48}$$

The estimates (3.10.44),(3.10.45), (3.10.46), (3.10.47) and (3.10.48) conclude the proof of Proposition 3.10.12. □

3.10.3 Estimates for $I \leq N-8$

Proposition 3.10.13. *We have for $I \leq N-8$*

$$\|w_1(q)^{\frac{1}{2}}\partial Z^I\widetilde{g}_2\|_{L^2} \leq C_0\varepsilon(1+t)^{C\varepsilon}, \tag{3.10.49}$$

$$\|\alpha w_1(q)^{\frac{1}{2}}\partial Z^I\widetilde{g}_2\|_{L^2} \leq C_0\varepsilon + C\varepsilon^{\frac{3}{2}}, \tag{3.10.50}$$

and for $I \leq N-9$

$$\|w(q)^{\frac{1}{2}}\partial Z^I\widetilde{g}_2\|_{L^2} \leq C_0\varepsilon(1+t)^{C\varepsilon}, \tag{3.10.51}$$

$$\|\alpha w(q)^{\frac{1}{2}}\partial Z^I\widetilde{g}_2\|_{L^2} \leq C_0\varepsilon + C\varepsilon^{\frac{3}{2}}. \tag{3.10.52}$$

This is a consequence of the following two propositions.

Proposition 3.10.14. *We have for $I \leq N-8$*

$$\frac{d}{dt}\sum_{J\leq I}\|w_1(q)^{\frac{1}{2}}\partial Z^J\widetilde{g}_2\|_{L^2}^2 + \sum_{J\leq I}\|w_1'(q)^{\frac{1}{2}}\bar{\partial}Z^J\widetilde{g}_2\|_{L^2}^2 \lesssim \frac{\varepsilon^3}{(1+t)^{1+\sigma}} + \frac{\varepsilon}{1+t}\sum_{J\leq I}\|w_1(q)^{\frac{1}{2}}\partial Z^J\widetilde{g}_2\|_{L^2}^2, \tag{3.10.53}$$

and

$$\frac{d}{dt}\sum_{J\leq I}\|\alpha w_1(q)^{\frac{1}{2}}\partial Z^J\widetilde{g}_2\|_{L^2}^2 + \sum_{J\leq I}\|\alpha w_1'(q)^{\frac{1}{2}}\bar{\partial}Z^J\widetilde{g}_2\|_{L^2}^2 \lesssim \frac{\varepsilon^3}{(1+t)^{1+\sigma}} + \varepsilon\|\alpha w_2'(q)^{\frac{1}{2}}\bar{\partial}Z^{I+1}\widetilde{g}_3\|_{L^2}^2. \tag{3.10.54}$$

Proposition 3.10.15. *We have for $I \leq N-9$*

$$\frac{d}{dt}\sum_{J\leq I}\|w_0(q)^{\frac{1}{2}}\partial Z^J\widetilde{g}_2\|_{L^2}^2+\sum_{J\leq I}\|w_0'(q)^{\frac{1}{2}}\bar{\partial}Z^J\widetilde{g}_2\|_{L^2}^2 \lesssim \frac{\varepsilon^3}{(1+t)^{1+\sigma}}++\frac{\varepsilon}{1+t}\sum_{J\leq I}\|w_0(q)^{\frac{1}{2}}\partial Z^J\widetilde{g}_2\|_{L^2}^2, \tag{3.10.55}$$

and

$$\frac{d}{dt}\sum_{J\leq I}\|\alpha w_0(q)^{\frac{1}{2}}\partial Z^J\widetilde{g}_2\|_{L^2}^2+\sum_{J\leq I}\|\alpha w_0'(q)^{\frac{1}{2}}\bar{\partial}Z^J\widetilde{g}_2\|_{L^2}^2 \lesssim \frac{\varepsilon^3}{(1+t)^{1+\sigma}}+\varepsilon\|\alpha w_1'(q)^{\frac{1}{2}}\bar{\partial}Z^{I+1}\widetilde{g}_2\|_{L^2}^2. \tag{3.10.56}$$

We assume Proposition 3.10.14 and 3.10.15 and prove Proposition 3.10.13.

Proof of Proposition 3.10.13. The inequalities (3.10.49) and (3.10.51) are straightforward consequences of (3.10.53) and (3.10.55). To prove (3.10.50), we integrate (3.10.54). We obtain

$$\begin{aligned}&\sum_{J\leq I}\|\alpha w_1(q)^{\frac{1}{2}}\partial Z^J\widetilde{g}_2\|_{L^2}^2+\int_0^t\sum_{J\leq I}\|\alpha w_1'(q)^{\frac{1}{2}}\bar{\partial}Z^J\widetilde{g}_2\|_{L^2}^2 d\tau\\ &\leq \sum_{J\leq I}\|\alpha w_1(q)^{\frac{1}{2}}\partial Z^J\widetilde{g}_2(0)\|_{L^2}^2+C\varepsilon^3+C\varepsilon\int_0^t\|\alpha w_2'(q)^{\frac{1}{2}}\bar{\partial}Z^{I+1}\widetilde{g}_3\|_{L^2}^2 d\tau.\end{aligned}$$

Thanks to Proposition 3.10.9, we have

$$\int_0^t\|\alpha w_2'(q)^{\frac{1}{2}}\bar{\partial}Z^{I+1}\widetilde{g}_3\|_{L^2}^2\lesssim\varepsilon^2,$$

and consequently

$$\sum_{J\leq I}\|\alpha w_1(q)^{\frac{1}{2}}\partial Z^J\widetilde{g}_2\|_{L^2}^2+\int_0^t\sum_{J\leq I}\|\alpha w_1'(q)^{\frac{1}{2}}\bar{\partial}Z^J\widetilde{g}_2\|_{L^2}^2 d\tau\leq C_0^2\varepsilon^2+C\varepsilon^3, \tag{3.10.57}$$

which proves (3.10.50). To prove (3.10.52), we integrate (3.10.56)

$$\begin{aligned}&\sum_{J\leq I}\|\alpha w_0(q)^{\frac{1}{2}}\partial Z^J\widetilde{g}_2\|_{L^2}^2+\int\sum_{J\leq I}\|\alpha w_0'(q)^{\frac{1}{2}}\bar{\partial}Z^J\widetilde{g}_2\|_{L^2}^2 d\tau\\ &\leq \sum_{J\leq I}\|\alpha w_0(q)^{\frac{1}{2}}\partial Z^J\widetilde{g}_2(0)\|_{L^2}^2+C\varepsilon^3+C\varepsilon\int_0^t\|\alpha w_1'(q)^{\frac{1}{2}}\bar{\partial}Z^{I+1}\widetilde{g}_2\|_{L^2}^2 d\tau.\end{aligned}$$

Thanks to (3.10.57), we have for $I\leq N-9$

$$\int_0^t\|\alpha w_1'(q)^{\frac{1}{2}}\bar{\partial}Z^{I+1}\widetilde{g}_2\|_{L^2}^2 d\tau\lesssim\varepsilon^2,$$

and consequently

$$\sum_{J\leq I}\|\alpha w_0(q)^{\frac{1}{2}}\partial Z^J\widetilde{g}_2\|_{L^2}^2\leq C_0^2\varepsilon^2+C\varepsilon^3,$$

which concludes the proof of Proposition 3.10.13. □

Proof of Proposition 3.10.14. It is sufficient to estimate the terms in the region $q<0$, since in the region $q>0$, we have $w_0=w_1=w_2$ so the estimates are strictly the same than in the previous section. Once again, the weight modulator α is used to tackle the crossed terms, which create a logarithmic loss in the estimates. However, in the region $q<0$, since $\alpha=1$, we write everything with the weight

w_1, and do everything as if no terms were present in the region $q>0$, since the influence of these terms have already been tackled.

We first estimate the term coming from the non commutation of the wave operator with the null decomposition,

$$\Upsilon\left(\frac{r}{t}\right)\frac{1}{r^2}\partial_\theta Z^I(h_0+\widetilde{h}).$$

Since $I+1\leq N-7$, we can use the Propositions 3.7.2 for $Z^{I+1}h_0$ and Proposition 3.7.5 for $Z^{I+1}\widetilde{h}$. We obtain

$$|Z^{I+1}(h_0+\widetilde{h})|\lesssim\frac{\varepsilon^2}{(1+|q|)^{\frac{1}{2}-\rho}}.$$

Therefore

$$\left\|\Upsilon\left(\frac{r}{t}\right)\frac{1}{r^2}\partial_\theta Z^I(h_0+\widetilde{h})\right\|_{L^2}\lesssim\frac{\varepsilon^2}{(1+t)^{1+\sigma}}\left\|\Upsilon\left(\frac{r}{t}\right)\frac{1}{(1+|q|)^{\frac{1}{2}-\rho}r^{1-\sigma}}\right\|_{L^2}\lesssim\frac{\varepsilon^2}{(1+t)^{1+\sigma}},\tag{3.10.58}$$

where we have used the calculation

$$\begin{aligned}\left\|\Upsilon\left(\frac{r}{t}\right)\frac{1}{(1+|q|)^{\frac{1}{2}-\rho}r^{1-\sigma}}\right\|_{L^2}^2&\leq 2\pi\int\Upsilon\left(\frac{r}{t}\right)^2\frac{1}{(1+|q|)^{1-2\rho}r^{2-2\sigma}}rdr\\&\leq 2\pi\int\frac{dq}{(1+|q|)^{2-2\rho-2\sigma}}<+\infty,\end{aligned}$$

if $\rho+\sigma<\frac{1}{2}$.

We now estimate $Z^I(\partial_U g_{LL}\partial_{\underline{L}}h)$. We have

$$\|\mathbb{1}_{q<0}w_1(q)^{\frac{1}{2}}Z^I(\partial_U g_{LL}\partial_{\underline{L}}h)\|_{L^2}\lesssim\sum_{I_1+I_2\leq I}\|\mathbb{1}_{q<0}w_1(q)^{\frac{1}{2}}\bar{\partial}Z^{I_1}g_{LL}\partial Z^{I_2}h\|_{L^2}$$

If $I_1\leq\frac{N}{2}$ we estimate

$$|\bar{\partial}Z^{I_1}g_{LL}|\lesssim\frac{1}{1+s}|Z^{I_1+1}g_{LL}|\lesssim\frac{\varepsilon(1+|q|)}{(1+s)^{\frac{5}{2}-\rho}}\lesssim\frac{(1+|q|)^{\rho+\sigma}}{(1+t)^{\frac{3}{2}+\sigma}}.$$

Therefore

$$\begin{aligned}\|\mathbb{1}_{q<0}w_1(q)^{\frac{1}{2}}\bar{\partial}Z^{I_1}g_{LL}\partial Z^{I_2}h\|_{L^2}&\lesssim\frac{\varepsilon}{(1+t)^{\frac{3}{2}+\sigma}}\left\|\mathbb{1}_{q<0}\frac{(1+|q|)^{\rho+\sigma}}{(1+|q|)^{\frac{1}{4}}}\partial Z^{I_2}h\right\|_{L^2}\\&\lesssim\frac{\varepsilon}{(1+t)^{\frac{3}{2}+\sigma}}\|\mathbb{1}_{q<0}w_3(q)^{\frac{1}{2}}\partial Z^{I_2}h\|_{L^2}\end{aligned}$$

if $\rho+\sigma\leq\frac{1}{4}$, and consequently

$$\|\mathbb{1}_{q<0}w_1(q)^{\frac{1}{2}}\bar{\partial}Z^{I_1}g_{LL}\partial Z^{I_2}h\|_{L^2}\lesssim\frac{\varepsilon^2}{(1+t)^{1+\sigma}}.\tag{3.10.59}$$

If $I_2\leq\frac{N}{2}$ we estimate, thanks to (3.4.42)

$$|\partial Z^{I_2}h|\leq\frac{\varepsilon}{(1+|q|)^{\frac{3}{2}-\rho}}$$

therefore

$$\|\mathbb{1}_{q<0}w_1(q)^{\frac{1}{2}}\bar{\partial}Z^{I_1}g_{LL}\partial Z^{I_2}h\|_{L^2}\lesssim\varepsilon\left\|\frac{\mathbb{1}_{q<0}}{(1+|q|)^{\frac{1}{4}+\frac{3}{2}-\rho}}\bar{\partial}Z^{I_1}g_{LL}\right\|_{L^2}.$$

We estimate

$$|\bar\partial Z^{I_1} g_{LL}| \lesssim \frac{1}{1+s}|Z^{I_1+1}g_{LL}| \lesssim \frac{1}{(1+t)^{\frac12+\sigma}(1+|q|)^{\frac12-\sigma}}|Z^{I_1+1}g_{LL}|.$$

We obtain

$$\begin{aligned}\|\mathbb{1}_{q<0}w_1(q)^{\frac12}\bar\partial Z^{I_1}g_{LL}\partial Z^{I_2}h\|_{L^2} &\lesssim \frac{\varepsilon}{(1+t)^{\frac12+\sigma}}\left\|\frac{\mathbb{1}_{q<0}}{(1+|q|)^{2+\frac14-\rho-\sigma}}Z^{I_1+1}g_{LL}\right\|_{L^2}\\ &\lesssim \frac{\varepsilon}{(1+t)^{\frac12+\sigma}}\left\|\frac{\mathbb{1}_{q<0}}{(1+|q|)^{1+\frac14-\rho-\sigma}}\partial Z^{I_1+1}g_{LL}\right\|_{L^2}\\ &\lesssim \frac{\varepsilon}{(1+t)^{\frac12+\sigma}}\left\|\mathbb{1}_{q<0}w_2'(q)^{\frac12}\bar\partial Z^{I_1+1}\widetilde{g}_3\right\|_{L^2}\end{aligned}$$

where in the last inequality we have used the wave coordinate condition, and the fact that, since for $q<0$

$$w_2'(q) = \frac{1+2\mu}{(1+|q|)^{2+2\mu}},$$

we have

$$\frac{1}{(1+|q|)^{1+\frac14-\rho-\sigma}} \le w_2'(q)^{\frac12}$$

if $\sigma+\rho+\mu\le\frac14$. Therefore

$$\|\mathbb{1}_{q<0}w_1(q)^{\frac12}\bar\partial Z^{I_1}g_{LL}\partial Z^{I_2}h\|_{L^2}\|w_1(q)^{\frac12}\partial Z^I\widetilde{g}_2\| \lesssim \frac{\varepsilon}{(1+t)^{\sigma}}\|\mathbb{1}_{q<0}w_2'(q)^{\frac12}\bar\partial Z^{I_1+1}\widetilde{g}_3\|_{L^2}^2 + \frac{\varepsilon^3}{(1+t)^{1+\sigma}}. \tag{3.10.60}$$

In view of (3.10.58), (3.10.59) and (3.10.60), we conclude the proof of Proposition 3.10.14. ☐

Proof of Proposition 3.10.15. We have already proved

$$\left\|w(q)^{\frac12}\Upsilon\left(\frac{r}{t}\right)\frac{1}{r^2}\partial_\theta Z^I(h_0+\widetilde{h})\right\|_{L^2} \lesssim \frac{\varepsilon^2}{(1+t)^{1+\sigma}}. \tag{3.10.61}$$

We now estimate $Z^I(\partial_U g_{LL}\partial_{\underline{L}}h)$. We have

$$\|w(q)^{\frac12}Z^I(\partial_U g_{LL}\partial_{\underline{L}}h)\|_{L^2} \lesssim \sum_{I_1+I_2\le I}\|w(q)^{\frac12}\bar\partial Z^{I_1}g_{LL}\partial Z^{I_2}h\|_{L^2}.$$

If $I_1\le\frac{N}{2}$ we use the estimate

$$|\bar\partial Z^{I_1}g_{LL}| \lesssim \frac{1}{1+s}|Z^{I_1+1}g_{LL}| \lesssim \frac{\varepsilon(1+|q|)}{(1+s)^{\frac52-\rho}} \lesssim \varepsilon\frac{(1+|q|)^{\rho+\sigma}}{(1+t)^{\frac32+\sigma}}.$$

Instead of estimating $\|w(q)^{\frac12}\bar\partial Z^{I_1}g_{LL}\partial Z^{I_2}h\|_{L^2}$ we estimate

$$\|w(q)^{\frac12}\bar\partial Z^{I_1}g_{LL}\partial Z^{I_2}(h_0+\widetilde{h})\|_{L^2} \textit{ and } \|w(q)^{\frac12}\bar\partial Z^{I_1}g_{LL}\partial Z^{I_2}\widetilde{g}_2|_{L^2}.$$

We can also estimate since $I_2+1\le N-7$, thanks to (3.4.19) and (3.4.21)

$$\left|\partial Z^{I_2}(h_0+\widetilde{h})\right| \lesssim \frac{\varepsilon}{(1+|q|)^{\frac32-\rho}}.$$

Therefore

$$\begin{aligned}\|\mathbb{1}_{q<0}w_0(q)^{\frac12}\bar\partial Z^{I_1}g_{LL}\partial Z^{I_2}h\|_{L^2} &\lesssim \varepsilon^2\left\|\mathbb{1}_{q<0}\Upsilon\left(\frac{r}{t}\right)\frac{(1+|q|)^{\rho+\sigma}}{(1+s)^{\frac32+\sigma}(1+|q|)^{\frac32-\rho}}\right\|_{L^2}\\ &\lesssim \frac{\varepsilon^2}{(1+t)^{1+\sigma}}\left\|\mathbb{1}_{q<0}\frac{1}{\sqrt{1+s}(1+|q|)^{\frac32-2\rho-\sigma}}\right\|_{L^2}\end{aligned}$$

and consequently

$$\|w_0(q)^{\frac{1}{2}}\bar{\partial}Z^{I_1}g_{LL}\partial Z^{I_2}h\|_{L^2}\lesssim\frac{\varepsilon^2}{(1+t)^{1+\sigma}}. \tag{3.10.62}$$

We estimate also

$$\|w_0(q)^{\frac{1}{2}}\bar{\partial}Z^{I_1}g_{LL}\partial Z^{I_2}\widetilde{g}_2\|_{L^2}\lesssim\frac{\varepsilon}{(1+t)^{\frac{3}{2}-\rho}}\|w_0(q)^{\frac{1}{2}}\partial Z^{I_2}\widetilde{g}_2\|_{L^2}\lesssim\frac{\varepsilon^2}{(1+t)^{\frac{3}{2}-\rho}}. \tag{3.10.63}$$

If $I_2\leq\frac{N}{2}$ we estimate, thanks to (3.4.42)

$$|\partial Z^{I_2}h|\leq\frac{\varepsilon}{(1+|q|)^{\frac{3}{2}-\rho}}$$

therefore

$$\begin{aligned}\|\mathbb{1}_{q<0}w_0(q)^{\frac{1}{2}}\bar{\partial}Z^{I_1}g_{LL}\partial Z^{I_2}h\|_{L^2}&\lesssim\varepsilon\left\|\frac{\mathbb{1}_{q<0}}{(1+|q|)^{\frac{3}{2}-\rho}}\bar{\partial}Z^{I_1}g_{LL}\right\|_{L^2}\\&\lesssim\frac{\varepsilon}{(1+t)^{\frac{1}{2}+\sigma}}\left\|\frac{\mathbb{1}_{q<0}}{(1+|q|)^{2-\rho-\sigma}}Z^{I_1+1}g_{LL}\right\|_{L^2}\\&\lesssim\frac{\varepsilon}{(1+t)^{\frac{1}{2}+\sigma}}\left\|\frac{\mathbb{1}_{q<0}}{(1+|q|)^{1-\rho-\sigma}}\partial Z^{I_1+1}g_{LL}\right\|_{L^2}\\&\lesssim\frac{\varepsilon}{(1+t)^{\frac{1}{2}+\sigma}}\|w_1'(q)^{\frac{1}{2}}\bar{\partial}Z^{I_1+1}\widetilde{g}_2\|_{L^2}\end{aligned}$$

where in the last inequality we have used the wave coordinate condition, and the fact that, since for $q<0$

$$w_1'(q)=\frac{1}{2(1+|q|)^{\frac{3}{2}}},$$

we have

$$\frac{1}{(1+|q|)^{1-\rho-\sigma}}\lesssim w_1'(q)^{\frac{1}{2}},$$

if $\sigma+\rho\leq\frac{1}{4}$. and $q<0$. Therefore

$$\|\mathbb{1}_{q<0}w_0(q)^{\frac{1}{2}}\bar{\partial}Z^{I_1}g_{LL}\partial Z^{I_2}h\|_{L^2}\|w_0(q)^{\frac{1}{2}}\partial Z^{I}\widetilde{g}_2\|\lesssim\varepsilon\|\mathbb{1}_{q<0}w_1'(q)^{\frac{1}{2}}\bar{\partial}Z^{I_1+1}\widetilde{g}_2\|^2_{L^2}+\frac{\varepsilon^3}{(1+t)^{1+2\sigma}}. \tag{3.10.64}$$

The estimates (3.10.61), (3.10.62), (3.10.63) and (3.10.64) conclude the proof of Proposition 3.10.15. □

3.11 Improvement of the estimates for Πb

In order to conclude the proof of Theorem 3.1.12, it still remains to ameliorate the bootstrap assumptions (3.4.4) and (3.4.5). To this end, we will set

$$\widetilde{b}^{(2)}(\theta)=\Pi\int_{\Sigma_{T,\theta}}(\partial_q\varphi)^2rdq. \tag{3.11.1}$$

Proposition 3.11.1. *We assume that the time T satisfies*

$$T\leq\exp\left(\frac{C}{\sqrt{\varepsilon}}\right).$$

There exists $(\varphi^{(2)}, g^{(2)})$ *solution of* (3.1.1) *in* $[0,T]$ *in the generalized wave coordinates* $H_{b^{(2)}}$*, such that, if we write* $g^{(2)} = g_{b^2} + \widetilde{g}$*, then* $(\varphi^{(2)}, \widetilde{g}^{(2)})$ *satisfies the same estimate as* $(\varphi, \widetilde{g})$*, and we have the estimates for* $b^{(2)}$

$$\left\| \partial_\theta^I \left(\Pi b^{(2)}(\theta) + \Pi \int_{\Sigma_{T,\theta}} (\partial_q \varphi^{(2)})^2 r dq \right) \right\|_{L^2} \leq C \frac{\varepsilon^4}{\sqrt{T}}, \text{ for } I \leq N-4,$$
$$\|\partial_\theta^I b(\theta)\|_{L^2} \leq 2C_0^2 \varepsilon^2, \text{ for } I \leq N.$$

The rest of this section is devoted to the proof of Proposition 3.11.1.

We solve the constraint equations with parameter $\widetilde{b}^{(2)}$. The initial data we obtain, constructed in Theorem 3.1.3 are of the form

$$g = g_{b^{(2)}} + \widetilde{g}^{(2)}$$

where we write

$$b^{(2)} = \widetilde{b}^{(2)} + b_0^{(2)} + b_1^{(2)} \cos(\theta) + b_2^{(2)} \sin(\theta),$$

with $b_0^{(2)}, b_1^{(2)}, b_2^{(2)}$ given by Theorem 3.1.3. We have the following estimates for the initial data at $t = 0$

$$\|\widetilde{g} - \widetilde{g}^{(2)}\|_{H^{N-3}_\delta} + \|\partial_t \widetilde{g} - \partial_t \widetilde{g}^{(2)}\|_{H^{N-4}_{\delta+1}} \lesssim \|\widetilde{b} - \widetilde{b}^{(2)}\|_{W^{N-4,2}} \leq C_0' \frac{\varepsilon^2}{\sqrt{T}},$$

thanks to (3.4.4), and

$$\|\widetilde{g} - \widetilde{g}^{(2)}\|_{H^{N+1}_\delta} + \|\partial_t \widetilde{g} - \partial_t \widetilde{g}^{(2)}\|_{H^{N}_{\delta+1}} \lesssim \varepsilon^2.$$

We solve, on an interval $[0, T_2]$, the system (3.2.4) in generalized coordinates given by $g_{b^{(2)}}$. We note $(\varphi^{(2)}, \widetilde{g}^{(2)})$ the solution.

We want to estimate the difference between $(\varphi^{(2)}, \widetilde{g}^{(2)})$ and $(\varphi, \widetilde{g})$. However, it will not be possible to estimate the difference with the same norms than when we estimated φ and $\widetilde{g}$. When we estimated h_0 we were able to use the condition

$$\left| \widetilde{b} + \Pi \int_{\Sigma_{T,\theta}} (\partial_q \varphi)^2 \right| \lesssim \frac{\varepsilon^2}{\sqrt{T}},$$

to obtain decay in $\frac{1}{\sqrt{1+|q|}}$ for h_0. However we want to keep the factor $\frac{1}{\sqrt{T}}$ in the estimates of the difference. To this end, we will loose the decay of $h_0 - h_0^{(2)}$ in $\frac{1}{\sqrt{1+|q|}}$ and consequently in $\widetilde{g} - \widetilde{g}^{(2)}$.

We will prove Proposition 3.11.1 with a bootstrap argument.

3.11.1 Bootstrap assumptions for $\varphi^{(2)} - \varphi$ and $\widetilde{g}^{(2)} - \widetilde{g}$

L^∞ estimates First some L^∞ estimates on $\varphi - \varphi^{(2)}$.

$$|Z^I(\varphi - \varphi^{(2)})| \leq \frac{2C_0\varepsilon^2}{\sqrt{T}\sqrt{1+s}(1+q)^{\frac{1}{2}-2\kappa-5\rho}}, \text{ for } I \leq N-20, \tag{3.11.2}$$

$$|Z^I(\varphi - \varphi^{(2)})| \leq \frac{2C_0\varepsilon^2}{\sqrt{T}(1+s)^{\frac{1}{2}-2\kappa-2\rho}}, \text{ for } I \leq N-18. \tag{3.11.3}$$

We use the decompositions

$$g^{(2)} = g_{b^{(2)}} + \Upsilon\left(\frac{r}{t}\right)(h_0^{(2)} + \widetilde{h}^{(2)})dq^2 + \widetilde{g}_2^{(2)}, \tag{3.11.4}$$

where $h_0^{(2)}$ satisfies the transport equation

$$\begin{cases} \partial_q h_0^{(2)} = -2r\left(\partial_q \varphi^{(2)}\right)^2 - 2b^{(2)}(\theta)\partial_q^2(\chi(q)q), \\ h_0^{(2)}|_{t=0} = 0, \end{cases}$$

and $\widetilde{h}^{(2)}$ satisfies the linear wave equation

$$\begin{cases} \Box\widetilde{h}^{(2)} = \Box h_0^{(2)} + g_{LL}^{(2)}\partial_q^2 h_0^{(2)} + 2\left(\partial_q\varphi^{(2)}\right)^2 - 2(R_{b^{(2)}})_{qq} + \widetilde{Q}_{\underline{LL}}(h_0^{(2)},\widetilde{g}^{(2)}), \\ (\widetilde{h}^{(2)},\partial_t\widetilde{h}^{(2)})|_{t=0} = (0,0), \end{cases}$$

We assume the following estimates on $h_0 - h_0^{(2)}$ for $I \leq N-12$

$$|Z^I(h_0 - h_0^{(2)})| \leq 2C_0\frac{\varepsilon^2}{\sqrt{T}}. \tag{3.11.5}$$

We introduce the two weight modulators

$$\begin{cases} \beta_1(q) = 1, \; q > 0, \\ \beta_1(q) = \frac{1}{(1+|q|)^\kappa}, \; q < 0, \end{cases}$$

and

$$\begin{cases} \beta_2(q) = 1, \; q > 0, \\ \beta_2(q) = \frac{1}{(1+|q|)^{2\kappa}}, \; q < 0, \end{cases}$$

with $0 < \kappa \ll 1$. We assume for $I \leq N-15$

$$\left\|\beta_1 w_0^{\frac{1}{2}}\partial Z^I(\widetilde{g}_2 - \widetilde{g}_2^{(2)})\right\|_{L^2} \leq \frac{2C_0\varepsilon^2}{\sqrt{T}}(1+t)^\rho \tag{3.11.6}$$

$$\left\|\alpha\beta_1 w^{\frac{1}{2}}\partial Z^I(\widetilde{g}_2 - \widetilde{g}_2^{(2)})\right\|_{L^2} \leq \frac{2C_0\varepsilon^2}{\sqrt{T}} \tag{3.11.7}$$

and

$$\left\|\beta_2 w_1^{\frac{1}{2}}\partial Z^{N-14}\left(\widetilde{g}_2 - \widetilde{g}_2^{(2)}\right)\right\|_{L^2} \leq \frac{2C_0\varepsilon^2}{\sqrt{T}}(1+t)^\rho \tag{3.11.8}$$

$$\left\|\alpha\beta_2 w_1^{\frac{1}{2}}\partial Z^{N-14}\left(\widetilde{g}_2 - \widetilde{g}_2^{(2)}\right)\right\|_{L^2} \leq \frac{2C_0\varepsilon^2}{\sqrt{T}}. \tag{3.11.9}$$

We use the decomposition

$$g^{(2)} = g_{b^{(2)}} + \Upsilon\left(\frac{r}{t}\right)h^{(2)}dq^2 + \widetilde{g}_3^{(2)}, \tag{3.11.10}$$

where $h^{(2)}$ is the solution of

$$\begin{cases} \Box_{g^{(2)}} h^{(2)} = -2(\partial_q\varphi^{(2)})^2 + 2(R_{b^{(2)}})_{qq} + Q_{\underline{LL}}(h^{(2)},\widetilde{g}^{(2)}), \\ (h^{(2)},\partial_t h^{(2)})|_{t=0} = (0,0). \end{cases}$$

We assume for $I \leq N-6$

$$\left\|\alpha\beta_2 w_0^{\frac{1}{2}}\partial Z^I\left(\varphi - \varphi^{(2)}\right)\right\|_{L^2} + \left\|\alpha\beta_2 w_2^{\frac{1}{2}}\partial Z^I\left(\widetilde{g}_3 - \widetilde{g}_3^{(2)}\right)\right\|_{L^2} + \frac{1}{\sqrt{1+t}}\left\|\alpha\beta_2 w_3^{\frac{1}{2}}\partial Z^I\left(h - h^{(2)}\right)\right\|_{L^2} \leq \frac{2C_0\varepsilon^2}{\sqrt{T}}, \tag{3.11.11}$$

and for $I \leq N-5$

$$\left\|\alpha\beta_2 w_0^{\frac{1}{2}}\partial Z^I\left(\varphi - \varphi^{(2)}\right)\right\|_{L^2} + \left\|\alpha\beta_2 w_2^{\frac{1}{2}}\partial Z^I\left(\widetilde{g}_3 - \widetilde{g}_3^{(2)}\right)\right\|_{L^2} + \frac{1}{\sqrt{1+t}}\left\|\alpha\beta_2 w_3^{\frac{1}{2}}\partial Z^I\left(h - h^{(2)}\right)\right\|_{L^2} \leq \frac{2C_0\varepsilon^2}{\sqrt{T}}(1+t)^\rho. \tag{3.11.12}$$

We use the decomposition

$$g^{(2)} = g_{b^{(2)}} + \Upsilon\left(\frac{r}{t}\right)h^{(2)}dq^2 + \Upsilon\left(\frac{r}{t}\right)k^{(2)}rdqd\theta + \widetilde{g}_4^{(2)},$$

where $k^{(2)}$ is the solution of

$$\begin{cases} \Box_g k^{(2)} = \partial_U g_{LL}^{(2)} \partial_q h^{(2)}, \\ (h^{(2)}, \partial_t h^{(2)})|_{t=0} = (0,0). \end{cases}$$

We assume for $I \leq N-4$

$$\begin{aligned} &\left\| \alpha_2 \beta_2 w_0^{\frac{1}{2}} \partial Z^I \left(\varphi - \varphi^{(2)} \right) \right\|_{L^2} + \left\| \alpha_2 \beta_2 w_2^{\frac{1}{2}} \partial Z^I \left(\widetilde{g}_3 - \widetilde{g}_3^{(2)} \right) \right\|_{L^2} \\ &+ \frac{1}{\sqrt{1+t}} \left\| \alpha_2 \beta_2 w_3^{\frac{1}{2}} \partial Z^I \left(h - h^{(2)} \right) \right\|_{L^2} + \frac{1}{\sqrt{1+t}} \left\| \alpha_2 \beta_2 w_3^{\frac{1}{2}} \partial Z^I \left(k - k^{(2)} \right) \right\|_{L^2} \leq \frac{2C_0 \varepsilon^2}{\sqrt{T}} (1+t)^\rho. \end{aligned} \tag{3.11.13}$$

To improve the estimates, we follow the same steps than when we ameliorated the bootstrap assumptions of Section 3.4. The difference of our new bootstrap assumptions compared with the estimates of Section 3.4 is at worse a factor $\frac{\varepsilon\sqrt{1+|q|}}{\sqrt{T}}$ in the region $q<0$. In the region $q>0$ the decay is the same and we have won a factor $\frac{\varepsilon}{\sqrt{T}}$. Therefore we can restrict our study to the region $q<0$: we will perform our estimates as if no term was present in the region $q>0$. We will follow the same steps as before, but with much less details since the mechanisms are the same.

Remark 3.11.2. *As long as the bootstrap estimates for $\varphi^{(2)} - \varphi$ and $\widetilde{g}^{(2)} - \widetilde{g}$ are satisfied, $\varphi^{(2)}$ and $\widetilde{g}^{(2)}$ satisfy the same estimates as φ and $\widetilde{g}$.*

L^∞ estimates using the weighted Klainerman-Sobolev inequality The following estimates are a direct consequence of the bootstrap assumptions and the weighted Klainerman-Sobolev inequality. For $I \leq N-8$ we have

$$\left| \partial Z^I \left(\varphi^{(2)} - \varphi \right) \right| \lesssim \frac{\varepsilon^2}{\sqrt{T}\sqrt{1+t}(1+|q|)^{\frac{1}{2}-2\kappa}}, \tag{3.11.14}$$

$$\left| \partial Z^I \left(\widetilde{g}_3^{(2)} - \widetilde{g}_3 \right) \right| \lesssim \frac{\varepsilon^2 (1+|q|)^{\frac{1}{2}+\mu+2\kappa}}{\sqrt{T}\sqrt{1+s}}, \tag{3.11.15}$$

$$\left| \partial Z^I \left(h^{(2)} - h \right) \right| \lesssim \frac{\varepsilon^2}{\sqrt{T}(1+|q|)^{\frac{1}{2}-2\kappa}}, \tag{3.11.16}$$

and for $I \leq N-17$

$$\left| \partial Z^I \left(\widetilde{g}_2^{(2)} - \widetilde{g}_2 \right) \right| \lesssim \frac{\varepsilon^2 (1+|q|)^\kappa}{\sqrt{T}\sqrt{1+s}\sqrt{1+|q|}}. \tag{3.11.17}$$

3.11.2 Improvement of the estimate of $h_0 - h_0^{(2)}$ and $\widetilde{h}^{(2)} - \widetilde{h}_0$

Estimate of $h_0 - h_0^{(2)}$ The quantity $h_0 - h_0^{(2)}$ satisfies the transport equation

$$\begin{cases} \partial_q \left(h_0^{(2)} - h_0 \right) = -2r \left(\left(\partial_q \varphi^{(2)}\right)^2 - (\partial_q \varphi)^2 \right) - 2 \left(b^{(2)}(\theta) - b(\theta) \right) \partial_q^2 (\chi(q) q), \\ (h_0^{(2)} - h_0)|_{t=0} = 0. \end{cases}$$

We write this equation under the form

$$\partial_q \left(h_0^{(2)} - h_0 \right) = -2r \left(\partial_q \varphi^{(2)} + \partial_q \varphi \right) \left(\partial_q \varphi^{(2)} - \partial_q \varphi \right) - 2 \left(b^{(2)}(\theta) - b(\theta) \right) \partial_q^2 (\chi(q) q).$$

For $k+l \leq N-7$, $k \geq 1$, the equivalent of estimate (3.7.12), that we obtain using (3.11.2) and (3.11.14) to estimate $\partial(\varphi - \varphi(2))$ and (3.4.15) and (3.4.28) to estimate $\partial(\varphi + \varphi(2))$ corresponds to (3.7.12) multiplied by $\frac{\varepsilon\sqrt{1+|q|}}{\sqrt{T}}$.

$$\left| \partial_q^k \partial_\theta^l \left(h_0 - h_0^{(2)} \right) \right| \lesssim \frac{\varepsilon^3}{\sqrt{T}(1+|q|)^{k+\frac{1}{2}-4\rho}}.$$

We obtain the estimate for $k=0$ by integrating the previous one with respect to q. We obtain, for $l\leq N-8$

$$|\partial_\theta^l h_0|\lesssim \frac{\varepsilon^3}{\sqrt{T}}.$$

For $k+l+j\leq N-8$, $k\geq 1$ and $j\geq 1$ the equivalent of (3.7.17) is

$$\left|\partial_s^j\partial_q^k\partial_\theta^l\left(h_0-h_0^{(2)}\right)\right|\lesssim \frac{\varepsilon^3}{\sqrt{T}(1+s)^{j+\frac12}(1+|q|)^{k-4\rho}}.$$

Consequently we have proved that for $I\leq N-8$ we have

$$\left|Z^I\left(h_0-h_0^{(2)}\right)\right|\lesssim\frac{\varepsilon^3}{\sqrt{T}}. \tag{3.11.18}$$

Estimation of $\widetilde{h}^{(2)}-\widetilde{h}_0$ The quantity $\widetilde{h}^{(2)}-\widetilde{h}_0$ satisfies the linear equation

$$\begin{cases}\Box\left(\widetilde{h}^{(2)}-\widetilde{h}\right)=\Box\left(h_0^{(2)}-h_0\right)+2\left(\left(\partial_q\varphi^{(2)}\right)^2-(\partial_q\varphi)^2\right)-2(R_{b^{(2)}})_{qq}+2(R_b)_{qq}\\ \qquad\qquad +g_{LL}^{(2)}\partial_q^2h_0^{(2)}-g_{LL}\partial_q^2h_0+\widetilde{Q}_{\underline{LL}}(h_0^{(2)},\widetilde{g}^{(2)})-\widetilde{Q}_{\underline{LL}}(h_0,\widetilde{g}),\\ \left(\widetilde{h}^{(2)}-\widetilde{h},\partial_t\left(\widetilde{h}^{(2)}-\widetilde{h}\right)\right)|_{t=0}=(0,0).\end{cases}$$

Proceeding as for the estimate of (3.7.26), and in view of the bootstrap assumptions for $\varphi-\varphi(2)$ and $\widetilde{g}-\widetilde{g}^{(2)}$ we obtain the analogue of (3.7.26) for $\Box\left(Z^I\widetilde{h}^{(2)}-Z^I\widetilde{h}\right)$, where the corresponding right-hand side gets multiplied by $\frac{\varepsilon\sqrt{1+|q|}}{\sqrt{T}}$. We obtain, for $I\leq N-10$ and $q<0$

$$\left|\Box\left(Z^I\widetilde{h}^{(2)}-Z^I\widetilde{h}\right)\right|\lesssim\frac{\varepsilon^3}{\sqrt{T}(1+s)^{\frac32}(1+|q|)^{\frac12}}.$$

Therefore if we perform the weighted energy estimate we obtain

$$\frac{d}{dt}\left\|w^{\frac12}\partial\left(Z^I\widetilde{h}^{(2)}-Z^I\widetilde{h}\right)\right\|_{L^2}\lesssim\left\|\frac{\varepsilon^3}{\sqrt{T}(1+s)^{\frac32}(1+|q|)^{\frac12}}\right\|_{L^2}\lesssim\frac{\varepsilon^3\ln(1+t)}{\sqrt{T}(1+t)},$$

and therefore for $I\leq N-10$ we have

$$\left\|w_0^{\frac12}\partial\left(Z^I\widetilde{h}^{(2)}-Z^I\widetilde{h}\right)\right\|_{L^2}\lesssim\frac{\varepsilon^3}{\sqrt{T}}(1+t)^\rho. \tag{3.11.19}$$

The weighted Klainerman-Sobolev inequality yields, for $I\leq N-12$

$$\left|\partial\left(Z^I\widetilde{h}^{(2)}-Z^I\widetilde{h}\right)\right|\lesssim\frac{\varepsilon^3(1+t)^\rho}{\sqrt{T}\sqrt{1+s}\sqrt{1+|q|}}. \tag{3.11.20}$$

3.11.3 Improvement of the L^∞ estimate for $\varphi-\varphi^{(2)}$

We write the equation satisfied by $\varphi^{(2)}-\varphi$

$$\Box_g\left(\varphi-\varphi^{(2)}\right)=\left(\left(g^{(2)}\right)^{\alpha\beta}-g^{\alpha\beta}\right)\partial_\alpha\partial_\beta\varphi^{(2)}+\left(H_{b^{(2)}}-H_b\right)^\rho\partial_\rho\varphi^{(2)}.$$

We limit ourselves to the region $q<0$. We estimate for $I+J\leq N-20$

$$Z^I\left(g_{LL}^{(2)}-g_{LL}\right)Z^J\partial^2\varphi.$$

With the wave coordinate condition and the estimate (3.11.17), we obtain, for $I \le N-17$

$$\left|Z^I\left(g^{(2)}_{LL}-g_{LL}\right)\right| \lesssim \frac{\varepsilon^2(1+|q|)^{\frac{3}{2}+\kappa}}{(1+s)^{\frac{3}{2}}}. \tag{3.11.21}$$

Moreover we have, for $J \le N-20$ thanks to (3.4.15)

$$|Z^J\partial^2\varphi| \lesssim \frac{1}{(1+|q|)^2}|Z^{J+2}\varphi| \lesssim \frac{\varepsilon^2}{\sqrt{T}\sqrt{1+s}(1+|q|)^{\frac{5}{2}-4\rho}}.$$

Consequently

$$\left|Z^I\left(g^{(2)}_{LL}-g_{LL}\right)Z^J\partial^2\varphi\right| \lesssim \frac{\varepsilon^3}{\sqrt{T}(1+s)^2(1+|q|)^{1-4\rho-\kappa}} \lesssim \frac{\varepsilon^3}{\sqrt{T}(1+s)^{2-5\rho-\kappa}(1+|q|)^{1+\rho}}.$$

We now estimate for $I+J \le N-20$

$$Z^I g_{LL} Z^J\partial^2\left(\varphi-\varphi^{(2)}\right).$$

We have, thanks to (3.5.8) and (3.11.3)

$$|Z^I g_{LL}| \lesssim \frac{\varepsilon(1+|q|)}{(1+s)^{\frac{3}{2}-\rho}},$$

$$\left|Z^J\partial^2\left(\varphi-\varphi^{(2)}\right)\right| \lesssim \frac{1}{(1+|q|)^2}\left|Z^{J+2}\left(\varphi-\varphi^{(2)}\right)\right| \lesssim \frac{\varepsilon^2}{\sqrt{T}(1+s)^{\frac{1}{2}-2\rho-2\kappa}(1+|q|)^2}.$$

Consequently

$$\left|Z^I g_{LL}Z^J\partial^2\left(\varphi-\varphi^{(2)}\right)\right| \lesssim \frac{\varepsilon^3}{(1+s)^{2-5\rho-2\kappa}(1+|q|)^{1+\rho}}$$

and the $L^\infty - L^\infty$ estimate yields for $I \le N-20$, since the initial data for $\varphi-\varphi^{(2)}$ are zero.

$$\left|Z^I\left(\varphi-\varphi^{(2)}\right)\right| \le \frac{C\varepsilon^3}{\sqrt{T}(1+s)^{\frac{1}{2}}(1+|q|)^{\frac{1}{2}-5\rho-2\kappa}}. \tag{3.11.22}$$

We now estimate for $I+J \le N-18$, thanks to (3.11.21) an (3.4.15) for the first inequality, and (3.5.8) and (3.11.14) for the second inequality

$$\left|Z^I\left(g^{(2)}_{LL}-g_{LL}\right)Z^J\partial^2\varphi\right| \lesssim \left(\frac{\varepsilon^2(1+|q|)^{\frac{3}{2}+\kappa}}{\sqrt{T}(1+s)^{\frac{3}{2}}}\right)\left(\frac{\varepsilon}{(1+|q|)^{\frac{5}{2}-4\rho}(1+s)^{\frac{1}{2}}}\right) \lesssim \frac{\varepsilon^3}{\sqrt{T}(1+s)^{2-5\rho-\kappa}(1+|q|)^{1+\rho}},$$

$$\left|Z^I g_{LL}Z^J\partial^2\left(\varphi-\varphi^{(2)}\right)\right| \lesssim \left(\frac{\varepsilon(1+|q|)}{(1+s)^{\frac{3}{2}-\rho}}\right)\left(\frac{\varepsilon^2(1+|q|)^{2\kappa}}{\sqrt{T}(1+|q|)^{\frac{3}{2}}\sqrt{1+s}}\right) \lesssim \frac{\varepsilon^3}{\sqrt{T}(1+s)^{2-\rho}(1+|q|)^{\frac{1}{2}-2\kappa}}.$$

Consequently, for $I \le N-18$ and $q<0$ we have

$$\left|\Box Z^I\left(\varphi-\varphi^{(2)}\right)\right| \lesssim \frac{\varepsilon^3}{\sqrt{T}(1+s)^{\frac{3}{2}-\rho-2\kappa}(1+|q|)}$$

and Lemma 3.7.6 yields, for $I \le N-18$, since the initial data for $\varphi-\varphi^{(2)}$ are zero.

$$\left|Z^I\left(\varphi-\varphi^{(2)}\right)\right| \le \frac{C\varepsilon^3}{\sqrt{T}(1+s)^{\frac{1}{2}-2\rho-2\kappa}}. \tag{3.11.23}$$

3.11.4 L^2 estimates

L^2 estimate for $\partial Z^I\left(\widetilde{g}_2^{(2)} - \widetilde{g}_2\right)$ with $I \leq N-15$ We have

$$\Box_g\left((\widetilde{g}_2)_{\mu\nu} - \left(\widetilde{g}_2^{(2)}\right)_{\mu\nu}\right) = f_{\mu\nu},$$

where the terms in $f_{\mu\nu}$ are

- the terms coming from the non commutation of the null decomposition with the wave operator: it is sufficient to study the term $\Upsilon\left(\frac{r}{t}\right)\frac{1}{r^2}\partial_\theta\left(h_0^{(2)} - h_0 + \widetilde{h}^{(2)} - \widetilde{h}\right)$,
- the semi-linear terms: it is sufficient to study $\partial_{\underline{L}}(h)\partial_U\left(g_{LL} - g_{LL}^{(2)}\right)$ and $\partial_U g_{LL}\partial_{\underline{L}}\left(h^{(2)} - h\right)$,
- the quasilinear terms: it is sufficient to study the terms $g_{LL}\partial_{\underline{L}}^2\left(\widetilde{g}^{(2)} - \widetilde{g}\right)$ and $\left(g_{LL}^{(2)} - g_{LL}\right)\partial_{\underline{L}}^2\widetilde{g}$,
- the crossed terms: they do not occur in the region $q<0$.

We estimate the first term. Thanks to (3.11.20) and (3.11.18) we have, for $I \leq N-15$

$$\left|\partial_\theta Z^I\left(h_0^{(2)} - h_0 + \widetilde{h}^{(2)} - \widetilde{h}\right)\right| \lesssim \frac{\varepsilon^3(1+|q|)^\rho}{\sqrt{T}}.$$

Therefore, we can estimate in the region $q<0$,

$$\left\|\beta_1 w_0^{\frac12}\Upsilon\left(\frac{r}{t}\right)\frac{1}{r^2}\partial_\theta Z^I\left(h_0^{(2)} - h_0 + \widetilde{h}^{(2)} - \widetilde{h}\right)\right\|_{L^2} \lesssim \left\|\Upsilon\left(\frac{r}{t}\right)\frac{\varepsilon^3}{\sqrt{T}(1+s)^2(1+|q|)^{\kappa-\rho}}\right\|_{L^2}$$

and consequently

$$\left\|\beta_1 w_0^{\frac12}\Upsilon\left(\frac{r}{t}\right)\frac{1}{r^2}\partial_\theta Z^I\left(h_0^{(2)} - h_0 + \widetilde{h}^{(2)} - \widetilde{h}\right)\right\|_{L^2} \lesssim \frac{\varepsilon^3}{\sqrt{T}(1+t)^{1+\kappa-\rho}}. \tag{3.11.24}$$

We now estimate the semi-linear terms. For $I \leq N-13$, we have, thanks to (3.4.43)

$$|\partial_{\underline{L}}(Z^I h)| \lesssim \frac{\varepsilon}{(1+|q|)^{\frac32-2\rho}},$$

Therefore we can estimate, for $I+J \leq N-15$ in the region $q<0$

$$\begin{aligned}
\left\|\beta_1 w_0^{\frac12} Z^I\partial_{\underline{L}}(h)Z^J\partial_U\left(g_{LL} - g_{LL}^{(2)}\right)\right\|_{L^2} &\lesssim \left\|\frac{\varepsilon}{(1+|q|)^{\frac32-2\rho+\kappa}(1+s)}Z^{J+1}\left(g_{LL} - g_{LL}^{(2)}\right)\right\|_{L^2}\\
&\lesssim \left\|\frac{\varepsilon}{(1+|q|)^{\frac12-2\rho+\kappa}(1+s)}\partial Z^{J+1}\left(g_{LL} - g_{LL}^{(2)}\right)\right\|_{L^2}\\
&\lesssim \frac{\varepsilon}{(1+t)^{\frac12+\sigma}}\left\|\frac{1}{(1+|q|)^{1-2\rho+\kappa-\sigma}}\bar\partial Z^{J+1}\left(\widetilde{g}_2 - \widetilde{g}_2^{(2)}\right)\right\|_{L^2}
\end{aligned}$$

and consequently

$$\left\|\beta_1 w_0^{\frac12} Z^I\partial_{\underline{L}}(h)Z^J\partial_U\left(g_{LL} - g_{LL}^{(2)}\right)\right\|_{L^2} \lesssim \frac{\varepsilon}{(1+t)^{\frac12+\sigma}}\left\|\beta_2 w_1'(q)^{\frac12}\bar\partial Z^{J+1}\left(\widetilde{g}_2 - \widetilde{g}_2^{(2)}\right)\right\|_{L^2}, \tag{3.11.25}$$

where we have used the wave coordinate condition and the fact that, for $q<0$

$$\beta_2 w_1'(q)^{\frac12} = \frac{1}{4(1+|q|)^{34+2\kappa}} \geq \frac{1}{(1+|q|)^{1-2\rho+\kappa-\sigma}}.$$

For $I \leq N-14$ thanks to Proposition 3.8.6, we have

$$|Z^I \partial_U g_{LL}| \lesssim \frac{\varepsilon(1+|q|)}{(1+s)^{\frac{5}{2}-2\rho}}.$$

In order to estimate

$$\left\|\beta_1 w_0^{\frac{1}{2}} \partial_U Z^I g_{LL} \partial_{\underline{L}} Z^J \left(h^{(2)}-h\right)\right\|_{L^2}$$

we will perform the estimates with $\left(h^{(2)}-h\right)$ replaced by $\left(h_0^{(2)}-h_0\right)$, $\left(\widetilde{h}^{(2)}-\widetilde{h}\right)$ and $\left(\widetilde{g}_2^{(2)}-\widetilde{g}_2\right)$. We estimate, in the region $q<0$, thanks to (3.11.18),

$$\begin{aligned}
\left\|\beta_1 w_0^{\frac{1}{2}} \partial_U Z^I g_{LL} \partial_{\underline{L}} Z^J \left(h_0^{(2)}-h_0\right)\right\|_{L^2} &\lesssim \frac{\varepsilon^3}{\sqrt{T}} \left\| \frac{\varepsilon(1+|q|)^{1-\kappa}}{(1+s)^{\frac{5}{2}-2\rho}(1+|q|)} \right\|_{L^2} \\
&\lesssim \frac{\varepsilon^3}{\sqrt{T}(1+t)^{\frac{3}{2}}},
\end{aligned}$$

thanks to (3.11.19)

$$\begin{aligned}
\left\|\beta_1 w_0^{\frac{1}{2}} \partial_U Z^I g_{LL} \partial_{\underline{L}} Z^J \left(\widetilde{h}^{(2)}-\widetilde{h}\right)\right\|_{L^2} &\lesssim \varepsilon \left\| \frac{\varepsilon(1+|q|)^{1-\kappa}}{(1+s)^{\frac{5}{2}-2\rho}} \partial_{\underline{L}} Z^J \left(\widetilde{h}^{(2)}-\widetilde{h}\right) \right\|_{L^2} \\
&\lesssim \frac{\varepsilon^3}{\sqrt{T}(1+t)^{\frac{3}{2}+\kappa-2\rho}},
\end{aligned}$$

and thanks to (3.11.6)

$$\begin{aligned}
\left\|\beta_1 w_0^{\frac{1}{2}} \partial_U Z^I g_{LL} \partial_{\underline{L}} Z^J \left(\widetilde{g}_2^{(2)}-\widetilde{g}_2\right)\right\|_{L^2} &\lesssim \varepsilon \left\| \frac{\varepsilon(1+|q|)^{1-\kappa}}{(1+s)^{\frac{5}{2}-2\rho}} \partial_{\underline{L}} Z^J \left(\widetilde{g}^{(2)}-\widetilde{g}\right) \right\|_{L^2} \\
&\lesssim \frac{\varepsilon^3}{\sqrt{T}(1+t)^{\frac{3}{2}-\rho}}.
\end{aligned}$$

Consequently, we have

$$\left\|\beta_1 w_0^{\frac{1}{2}} \partial_U Z^I g_{LL} \partial_{\underline{L}} Z^J \left(h^{(2)}-h\right)\right\|_{L^2} \lesssim \frac{\varepsilon^3}{\sqrt{T}(1+t)^{\frac{3}{2}-\rho}}. \tag{3.11.26}$$

The other terms are similar to estimate. Thanks to (3.11.24), (3.11.25) and (3.11.26), the energy inequality yields for $I \leq N-15$

$$\begin{aligned}
&\frac{d}{dt}\left\|\beta_1 w_0^{\frac{1}{2}} \partial Z^I \left(\widetilde{g}_2^{(2)}-\widetilde{g}_2\right)\right\|_{L^2}^2 + \left\|\beta_1 w_0'(q)^{\frac{1}{2}} \bar{\partial} Z^I \left(\widetilde{g}_2^{(2)}-\widetilde{g}_2\right)\right\|_{L^2}^2 \\
\lesssim &\frac{\varepsilon^3}{\sqrt{T}(1+t)^{1+\sigma}} \left\|\beta_1 w_0^{\frac{1}{2}} \partial Z^I \left(\widetilde{g}_2^{(2)}-\widetilde{g}_2\right)\right\|_{L^2} + \frac{\varepsilon}{(1+t)^{\sigma}} \left\|\beta_2 w_1'(q)^{\frac{1}{2}} \bar{\partial} Z^{I+1} \left(\widetilde{g}_2-\widetilde{g}_2^{(2)}\right)\right\|_{L^2}^2 \\
&+ \frac{\varepsilon}{(1+t)^{1+\sigma}} \left\|\beta_1 w_0^{\frac{1}{2}} \partial Z^I \left(\widetilde{g}_2^{(2)}-\widetilde{g}_2\right)\right\|_{L^2}^2.
\end{aligned} \tag{3.11.27}$$

L^2 estimate for $\partial Z^I \left(\widetilde{g}_2^{(2)}-\widetilde{g}_2\right)$ with $I \leq N-14$. We follow the same steps as in the previous paragraph. First we still have

$$\begin{aligned}
\left\|\beta_2 w_1^{\frac{1}{2}} \Upsilon\left(\frac{r}{t}\right) \frac{1}{r^2} \partial_\theta Z^I \left(h_0^{(2)}-h_0+\widetilde{h}^{(2)}-\widetilde{h}\right)\right\|_{L^2} &\lesssim \left\| \Upsilon\left(\frac{r}{t}\right) \frac{\varepsilon^3}{\sqrt{T}(1+s)^2(1+|q|)^{\frac{1}{4}+2\kappa-\rho}} \right\|_{L^2} \\
&\lesssim \frac{\varepsilon^3}{\sqrt{T}(1+t)^{\frac{5}{4}}}.
\end{aligned}$$

We estimate the second terms for $I+J\leq N-14$

$$\begin{aligned}\left\|\beta_2 w_1^{\frac12} Z^I \partial_{\underline{L}}(h) Z^J\partial_U\left(g_{LL}-g_{LL}^{(2)}\right)\right\|_{L^2} &\lesssim \left\|\frac{\varepsilon}{(1+|q|)^{\frac32-2\rho+\frac14+2\kappa}(1+s)}Z^{J+1}\left(g_{LL}-g_{LL}^{(2)}\right)\right\|_{L^2}\\ &\lesssim \left\|\frac{\varepsilon}{(1+|q|)^{\frac34-2\rho+2\kappa}(1+s)}\partial Z^{J+1}\left(g_{LL}-g_{LL}^{(2)}\right)\right\|_{L^2}\\ &\lesssim \frac{\varepsilon}{(1+t)^{\frac12+\sigma}}\left\|\frac{1}{(1+|q|)^{\frac54-2\rho+2\kappa-\sigma}}\bar\partial Z^{J+1}\left(\widetilde g_3-\widetilde g_3^{(2)}\right)\right\|_{L^2}\\ &\lesssim \frac{\varepsilon}{(1+t)^{\frac12+\sigma}}\left\|\beta_2 w_2'(q)^{\frac12}\bar\partial Z^{J+1}\left(\widetilde g_3-\widetilde g_3^{(2)}\right)\right\|_{L^2},\end{aligned}$$

where we have used the wave coordinate condition and the fact that

$$\beta_2 w_2'(q)^{\frac12}=\frac{1}{(1+|q|)^{1+2\kappa+\mu}}\geq\frac{1}{(1+|q|)^{\frac54-2\rho+2\kappa-\sigma}}.$$

The other terms are similar to estimate than for $I\leq N-15$. The energy inequality yields for $I\leq N-14$

$$\begin{aligned}&\frac{d}{dt}\left\|\beta_2 w_1^{\frac12}\partial Z^I\left(\widetilde g_2^{(2)}-\widetilde g_2\right)\right\|_{L^2}^2+\left\|\beta_2 w_1'(q)^{\frac12}\bar\partial Z^I\left(\widetilde g_2^{(2)}-\widetilde g_2\right)\right\|_{L^2}^2\\ \lesssim&\frac{\varepsilon^3}{\sqrt{T}(1+t)^{1+\sigma}}\left\|\beta_2 w_1^{\frac12}\partial Z^I\left(\widetilde g_2^{(2)}-\widetilde g_2\right)\right\|_{L^2}+\frac{\varepsilon}{(1+t)^{\sigma}}\left\|\beta_2 w_2'(q)^{\frac12}\bar\partial Z^{I+1}\left(\widetilde g_3-\widetilde g_3^{(2)}\right)\right\|_{L^2}^2\\ &+\frac{\varepsilon}{(1+t)^{1+\sigma}}\left\|\beta_2 w_1^{\frac12}\partial Z^I\left(\widetilde g_2^{(2)}-\widetilde g_2\right)\right\|_{L^2}^2.\end{aligned}\tag{3.11.28}$$

L^2 estimates for $\partial Z^I\left(\varphi^{(2)}-\varphi\right)$ with $I\leq N-6$. We estimate for $I+J\leq N-6$, $J\leq N-7$,

$$\left\|\beta_2 w_0^{\frac12}Z^I\left(g_{LL}^{(2)}-g_{LL}\right)\partial_q^2 Z^J\varphi\right\|_{L^2}.$$

If $I\leq\frac{N-7}{2}$ we can estimate, thanks to (3.11.21)

$$\left|Z^I\left(g_{LL}^{(2)}-g_{LL}\right)\right|\lesssim\frac{\varepsilon^2(1+|q|)^{\frac32+\kappa}}{\sqrt{T}(1+s)^{\frac32}},$$

and therefore, if we restrict our quantities to $q<0$

$$\begin{aligned}\left\|\beta_2 w_0^{\frac12}Z^I\left(g_{LL}^{(2)}-g_{LL}\right)\partial_q^2 Z^J\varphi\right\|_{L^2}&\lesssim\left\|\frac{\varepsilon^2(1+|q|)^{\frac32+\kappa}}{\sqrt{T}(1+s)^{\frac32}(1+|q|)^{1+2\kappa}}\partial Z^{J+1}\varphi\right\|_{L^2}\\ &\lesssim\frac{\varepsilon^2}{\sqrt{T}(1+t)^{1+\kappa}}\left\|w^{\frac12}\partial Z^{J+1}\varphi\right\|_{L^2}\\ &\lesssim\frac{\varepsilon^3}{\sqrt{T}(1+t)^{1+\kappa}}.\end{aligned}$$

The case $J\leq\frac{N-6}{2}$ can be treated as in Section 3.10.2.

We now evaluate

$$\left\|\beta_2 w_0^{\frac12}Z^I g_{LL}\partial_q^2 Z^J\left(\varphi^{(2)}-\varphi\right)\right\|_{L^2}$$

for $I+J\leq N-6$ and $J\leq\frac{N-6}{2}$. We have, since $\frac{N-6}{2}+2\leq N-20$

$$\left|\partial_q^2 Z^J\left(\varphi^{(2)}-\varphi\right)\right|\lesssim\frac{\varepsilon^2}{\sqrt{T}\sqrt{1+s}(1+|q|)^{\frac52-5\rho-2\kappa}}.$$

Therefore we can estimate

$$\begin{aligned}\left\|\beta_2 w_0^{\frac12} Z^I g_{LL}\partial_q^2 Z^J\left(\varphi^{(2)}-\varphi\right)\right\|_{L^2} &\lesssim \left\|\frac{\varepsilon^2}{\sqrt{T}\sqrt{1+s}(1+|q|)^{\frac52-5\rho}}Z^I g_{LL}\right\|_{L^2}\\ &\lesssim \left\|\frac{\varepsilon^2}{\sqrt{T}\sqrt{1+s}(1+|q|)^{\frac32-5\rho}}\partial Z^I g_{LL}\right\|_{L^2}\\ &\lesssim \left\|\frac{\varepsilon^2}{\sqrt{T}(1+s)^{\frac32}(1+|q|)^{\frac32-5\rho}}Z^{I+1}\widetilde{g}_3\right\|_{L^2}\\ &\lesssim \left\|\frac{\varepsilon^2}{\sqrt{T}(1+s)^{\frac32}(1+|q|)^{\frac12-5\rho}}\partial Z^{I+1}\widetilde{g}_3\right\|_{L^2}\\ &\lesssim \frac{\varepsilon^2}{\sqrt{T}(1+t)^{\frac32-5\rho-\mu}}\left\|w_2^{\frac12}\partial Z^{I+1}\widetilde{g}_3\right\|_{L^2}\lesssim \frac{\varepsilon^3}{\sqrt{T}(1+t)^{\frac32-5\rho-\mu}}.\end{aligned}$$

The case $I\leq \frac{N-6}{2}$ can be treated similarly than in Section 3.10.2. The weighted energy estimate yields

$$\frac{d}{dt}\left\|\beta_2 w_0^{\frac12}\partial Z^I\left(\varphi_2^{(2)}-\varphi\right)\right\|_{L^2}^2+\left\|\beta_2 w_0'(q)^{\frac12}\bar{\partial} Z^I\left(\varphi_2^{(2)}-\varphi\right)\right\|_{L^2}^2\lesssim \frac{\varepsilon^3}{\sqrt{T}(1+t)^{1+\kappa}}\left\|\beta_2 w^{\frac12}\partial Z^I\left(\varphi_2^{(2)}-\varphi\right)\right\|_{L^2}. \tag{3.11.29}$$

Consequently, since the initial data for $\varphi_2^{(2)}-\varphi$ are zero we have

$$\left\|\beta_2 w_0^{\frac12}\partial Z^I\left(\varphi_2^{(2)}-\varphi\right)\right\|_{L^2}\lesssim\frac{\varepsilon^3}{\sqrt{T}}. \tag{3.11.30}$$

L^2 estimates for $\partial Z^I\left(h^{(2)}-h\right)$ with $I\leq N-6$. We write the equation satisfied by $h^{(2)}-h$

$$\Box_g\left(h-h^{(2)}\right)=2(\partial_q\varphi^{(2)})^2-2(\partial_q\varphi)^2+2(R_b)_{qq}-2(R_{b^{(2)}})_{qq}+Q_{\underline{L}\underline{L}}(h,\widetilde{g})-Q_{\underline{L}\underline{L}}(h^{(2)},\widetilde{g}^{(2)}).$$

We first estimate for $I+J\leq N-6$ and $I\leq\frac{N-6}{2}$. We recall that we restrict all the quantities to $q<0$ (therefore $w_3=w_0$).

$$\begin{aligned}\left\|\beta_2 w_0^{\frac12}\partial_q Z^I\left(\varphi-\varphi^{(2)}\right)\partial_q Z^J\varphi\right\|_{L^2}&\lesssim\left\|\frac{\varepsilon^2}{\sqrt{T}\sqrt{1+s}(1+|q|)^{\frac32-5\rho}}\partial_q Z^J\varphi\right\|_{L^2}\\ &\lesssim\frac{\varepsilon^2}{\sqrt{T}\sqrt{1+t}}\|w_0^{\frac12}\partial_q Z^J\varphi\|_{L^2}\lesssim\frac{\varepsilon^3}{\sqrt{T}\sqrt{1+t}}.\end{aligned}$$

We now estimate the quasilinear term

$$\left\|\beta_2 w_0^{\frac12}Z^I\left(g_{LL}^{(2)}-g_{LL}\right)\partial_q^2 Z^J h\right\|_{L^2}$$

for $I+J\leq N-6$ and $I\leq\frac{N-6}{2}$. We have

$$\begin{aligned}\left\|\beta_2 w_0^{\frac12}Z^I\left(g_{LL}^{(2)}-g_{LL}\right)\partial_q^2 Z^J h\right\|_{L^2}&\lesssim\left\|\frac{\varepsilon^2(1+|q|)^{\frac32+\kappa}}{\sqrt{T}(1+s)^{\frac32}(1+|q|)^{1+2\kappa}}\partial Z^{J+1}h\right\|_{L^2}\\ &\lesssim\frac{\varepsilon^2}{\sqrt{T}(1+t)^{1+\kappa}}\left\|w_0^{\frac12}\partial Z^{J+1}h\right\|_{L^2}\\ &\lesssim\frac{\varepsilon^3}{\sqrt{T}(1+t)^{\frac12+\kappa}}.\end{aligned}$$

The other terms can be treated as in the proof of Proposition 3.10.11. The energy inequality yields

$$\frac{d}{dt}\left\|\beta_2 w_0^{\frac12}\partial Z^I\left(h_2^{(2)}-h\right)\right\|_{L^2}^2+\left\|\beta_2 w_0'(q)^{\frac12}\bar\partial Z^I\left(h_2^{(2)}-h\right)\right\|_{L^2}^2 \lesssim \frac{\varepsilon^3}{\sqrt{T}(1+t)^{\frac12}}\left\|\beta_2 w_0^{\frac12}\partial Z^I\left(h_2^{(2)}-h\right)\right\|_{L^2}.$$

Consequently, since the initial data for $h_2^{(2)}-h$ are zero, we have

$$\left\|\beta_2 w_0^{\frac12}\partial Z^I\left(h_2^{(2)}-h\right)\right\|_{L^2}\lesssim \frac{\varepsilon^3\sqrt{1+t}}{\sqrt{T}}. \tag{3.11.31}$$

L^2 estimates for $\partial Z^I\left(\widetilde g^{(2)}-\widetilde g\right)$ with $I\le N-6$. As usual we estimate the following contributions

- the terms coming from the non commutation of the null decomposition with the wave operator: it is sufficient to study the term $\Upsilon\left(\frac rt\right)\frac{1}{r^2}\partial_\theta\left(h^{(2)}-h\right)$,
- the semi-linear terms: it is sufficient to study $\partial_{\underline L}(h)\partial_U\left(g_{LL}-g_{LL}^{(2)}\right)$ and $\partial_U g_{LL}\partial_{\underline L}\left(h^{(2)}-h\right)$,
- the quasilinear terms: it is sufficient to study the terms $g_{LL}\partial_{\underline L}^2\left(\widetilde g^{(2)}-\widetilde g\right)$ and $\left(g_{LL}^{(2)}-g_{LL}\right)\partial_{\underline L}^2\widetilde g$.

We estimate the first term. We recall that we restrict all the quantities to $q<0$.

$$\begin{aligned}\left\|\beta_2 w_2^{\frac12}\chi\left(\frac rt\right)\frac{1}{r^2}\partial_\theta Z^I\left(h^{(2)}-h\right)\right\|_{L^2}&\lesssim\left\|\beta_2\frac{1}{(1+s)^2(1+|q|)^{\frac12+\mu}}Z^{I+1}\left(h^{(2)}-h\right)\right\|_{L^2}\\&\lesssim\left\|\beta_2\frac{1}{(1+s)^{\frac32+\sigma}(1+|q|)^{\mu-\sigma}}\partial Z^{I+1}\left(h^{(2)}-h\right)\right\|_{L^2}\\&\lesssim\frac{1}{(1+t)^{\frac32+\sigma}}\left\|\beta_2 w_0^{\frac12}\partial Z^{I+1}\left(h^{(2)}-h\right)\right\|_{L^2}.\end{aligned}$$

We estimate the second term

$$\left\|\beta_2 w_1^{\frac12}Z^I\partial_{\underline L}(h)Z^J\partial_U\left(g_{LL}-g_{LL}^{(2)}\right)\right\|_{L^2}$$

for $I+J\le N-6$ and $J\le\frac{N-6}{2}$. We have, thanks to (3.11.21)

$$\left|Z^J\partial_U\left(g_{LL}-g_{LL}^{(2)}\right)\right|\lesssim\frac{1}{1+s}\left|Z^{J+1}\left(g_{LL}-g_{LL}^{(2)}\right)\right|\lesssim\frac{\varepsilon^2(1+|q|)^{\frac32+\kappa}}{\sqrt{T}(1+s)^{\frac52}}.$$

Therefore we can estimate

$$\begin{aligned}\left\|\beta_2 w_2^{\frac12}Z^I\partial_{\underline L}(h)Z^J\partial_U\left(g_{LL}-g_{LL}^{(2)}\right)\right\|_{L^2}&\lesssim\left\|\frac{\varepsilon^2(1+|q|)^{\frac32+\kappa}}{\sqrt{T}(1+s)^{\frac52}(1+|q|)^{\frac12+\mu+2\kappa}}\partial_{\underline L}Z^I h\right\|_{L^2}\\&\lesssim\left\|\frac{\varepsilon^2}{\sqrt{T}(1+s)^{\frac32+\mu+\kappa}}\partial_{\underline L}Z^I h\right\|_{L^2}\\&\lesssim\frac{\varepsilon^2}{\sqrt{T}(1+t)^{\frac32+\mu+\kappa}}\|w^{\frac12}\partial_{\underline L}Z^I h\|_{L^2}\lesssim\frac{\varepsilon^3}{\sqrt{T}(1+t)^{1+\mu+\kappa}}.\end{aligned}$$

The other terms are treated as in the proof of Proposition 3.10.12. We have proved, when we restrict ourselves to $q<0$

$$\begin{aligned}&\frac{d}{dt}\left\|\beta_2 w_2^{\frac12}\partial Z^I\left(\widetilde g_3^{(2)}-\widetilde g_3\right)\right\|_{L^2}^2+\left\|\beta_2 w_2'(q)^{\frac12}\bar\partial Z^I\left(\widetilde g_3^{(2)}-\widetilde g_3\right)\right\|_{L^2}^2\\&\lesssim\Bigg(\frac{\varepsilon^3}{\sqrt{T}(1+t)^{1+\sigma}}+\frac{\varepsilon}{(1+t)^{\frac12+\sigma}}\left\|\beta_2 w_2'(q)^{\frac12}\bar\partial Z^{J+1}\left(\widetilde g_4-\widetilde g_4^{(2)}\right)\right\|_{L^2}\\&\quad+\frac{1}{(1+t)^{\frac32+\sigma}}\left\|\beta_2 w^{\frac12}\partial Z^{I+1}\left(h^{(2)}-h\right)\right\|_{L^2}\Bigg)\left\|\beta_2 w_2^{\frac12}\partial Z^I\left(\widetilde g_3^{(2)}-\widetilde g_3\right)\right\|_{L^2}.\end{aligned}\tag{3.11.32}$$

L^2 estimates for $I \leq N-4$ We can prove, following Section 3.10.1 that, since we do as if no quantity was present for $q>0$,

$$\begin{aligned}
&\frac{d}{dt}\Bigg(\left\| \beta_2 w^{\frac12} \partial Z^I \left(\varphi - \varphi^{(2)}\right)\right\|_{L^2}^2 + \left\| \beta_2 w_2^{\frac12} \partial Z^I \left(\widetilde{g}_4 - \widetilde{g}_4^{(2)}\right)\right\|_{L^2}^2 \\
&+ \frac{1}{\varepsilon(1+t)} \left\| \beta_2 w_3^{\frac12} \partial Z^I \left(h - h^{(2)}\right)\right\|_{L^2}^2 + \frac{1}{\varepsilon(1+t)} \left\| \beta_2 w_3^{\frac12} \partial Z^I \left(k - k^{(2)}\right)\right\|_{L^2}^2 \Bigg) \\
&+ \left\| \beta_2 w_0'(q)^{\frac12} \bar\partial Z^I \left(\varphi - \varphi^{(2)}\right)\right\|_{L^2}^2 + \left\| \beta_2 w_2'(q)^{\frac12} (q)' \bar\partial Z^I \left(\widetilde{g}_4 - \widetilde{g}_4^{(2)}\right)\right\|_{L^2}^2 \\
&+ \frac{1}{\varepsilon(1+t)} \left\| \beta_2 w_3'(q)^{\frac12} \bar\partial Z^I \left(h - h^{(2)}\right)\right\|_{L^2}^2 + \frac{1}{\varepsilon(1+t)} \left\| \beta_2 w_3'(q)^{\frac12} \bar\partial Z^I \left(k - k^{(2)}\right)\right\|_{L^2}^2 . \\
&\lesssim \frac{\sqrt{\varepsilon}}{1+t} \Bigg(\left\| \beta_2 w^{\frac12} \partial Z^I \left(\varphi - \varphi^{(2)}\right)\right\|_{L^2}^2 + \left\| \beta_2 w_2^{\frac12} \partial Z^I \left(\widetilde{g}_4 - \widetilde{g}_4^{(2)}\right)\right\|_{L^2}^2 \\
&+ \frac{1}{\varepsilon(1+t)} \left\| \beta_2 w_3^{\frac12} \partial Z^I \left(h - h^{(2)}\right)\right\|_{L^2}^2 + \frac{1}{\varepsilon(1+t)} \left\| \beta_2 w_3^{\frac12} \partial Z^I \left(k - k^{(2)}\right)\right\|_{L^2}^2 \Bigg) + O\left(\frac{\varepsilon^{\frac92}}{T(1+t)}\right)
\end{aligned} \tag{3.11.33}$$

3.11.5 Conclusion of the proof of Proposition 3.11.1

Estimate (3.11.18) gives us for $I \leq N-8$

$$\left| Z^I \left(h_0 - h_0^{(2)}\right)\right| \leq \frac{C\varepsilon^3}{\sqrt{T}}.$$

Estimate (3.11.22) gives us for $I \leq N-18$

$$\left| Z^I \left(\varphi - \varphi^{(2)}\right)\right| \leq \frac{C\varepsilon^3}{\sqrt{T}(1+s)^{\frac12}(1+|q|)^{\frac12 - 5\rho - 2\kappa}}.$$

Estimate (3.11.23) gives us for $I \leq N-16$

$$\left| Z^I \left(\varphi - \varphi^{(2)}\right)\right| \leq \frac{C\varepsilon^3}{\sqrt{T}(1+s)^{\frac12 - 2\rho - 2\kappa}}.$$

Therefore, if $C\varepsilon \leq C_0$ we have ameliorated the L^∞ estimates (3.11.5), (3.11.2) and (3.11.3). Estimate (3.11.33) implies, following the proof of Proposition 3.10.2,

$$\begin{aligned}
&\left\| \beta_2 w^{\frac12} \partial Z^I \left(\varphi - \varphi^{(2)}\right)\right\|_{L^2} + \left\| \beta_2 w_2^{\frac12} \partial Z^I \left(\widetilde{g}_3 - \widetilde{g}_3^{(2)}\right)\right\|_{L^2} \\
&+ \frac{1}{\sqrt{\varepsilon(1+t)}} \left\| \beta_2 w_3^{\frac12} \partial Z^I \left(h - h^{(2)}\right)\right\|_{L^2} + \frac{1}{\sqrt{\varepsilon(1+t)}} \left\| \beta_2 w_3^{\frac12} \partial Z^I \left(k - k^{(2)}\right)\right\|_{L^2} \\
&\leq \frac{1}{\sqrt{T}} \left(C_0\varepsilon^2 + \varepsilon^2\right)(1+t)^{C\sqrt{\varepsilon}}.
\end{aligned}$$

Therefore, if we had chosen $C_0 \geq 2$ and $C\sqrt{\varepsilon} \leq \rho$ we have ameliorated this estimate (3.11.13) and (3.11.12). Moreover we have

$$\left\| \beta_2 w_3^{\frac12} \partial Z^I \left(h - h^{(2)}\right)\right\|_{L^2} \lesssim \frac{\varepsilon^{\frac52}}{\sqrt{T}}(1+t)^{\frac12 + C\sqrt{\varepsilon}}.$$

Estimate (3.11.33) also implies

$$\int_0^t \frac{1}{(1+t)^\sigma}\left\|\beta_2 w_2'(q)^{\frac{1}{2}}\bar{\partial} Z^I\left(\widetilde{g}_4-\widetilde{g}_4^{(2)}\right)\right\|_{L^2}^2 \lesssim \frac{\varepsilon^4}{T}$$

and consequently, estimate (3.11.32), together with the bootstrap assumption (3.11.11) yields

$$\begin{aligned}&\frac{d}{dt}\left\|\beta_2 w_2^{\frac{1}{2}}\partial Z^I\left(\widetilde{g}_3^{(2)}-\widetilde{g}_3\right)\right\|_{L^2}^2+\left\|\beta_2 w_2'(q)^{\frac{1}{2}}\bar{\partial} Z^I\left(\widetilde{g}_3^{(2)}-\widetilde{g}_3\right)\right\|_{L^2}^2\\ \lesssim&\frac{\varepsilon^{\frac{9}{2}}}{T(1+t)^{1+\sigma}}+\frac{\varepsilon}{(1+t)^\sigma}\left\|(\beta_2 w_2^{\frac{1}{2}})'\bar{\partial} Z^{J+1}\left(\widetilde{g}_4-\widetilde{g}_4^{(2)}\right)\right\|_{L^2}^2,\end{aligned}$$

Therefore, when we integrate we obtain

$$\begin{aligned}&\left\|\beta_2 w_2^{\frac{1}{2}}\partial Z^I\left(\widetilde{g}_3^{(2)}-\widetilde{g}_3\right)\right\|_{L^2}^2+\int_0^t\left\|\left(\beta_2 w_2^{\frac{1}{2}}\right)'\bar{\partial} Z^I\left(\widetilde{g}_3^{(2)}-\widetilde{g}_3\right)\right\|_{L^2}^2\\ \lesssim&\frac{C_0^2\varepsilon^4}{T}+C^2\frac{\varepsilon^{\frac{9}{2}}}{T}.\end{aligned}$$

Therefore, for $C\varepsilon^{\frac{1}{2}}\le\frac{C_0}{2}$, this, together with (3.11.31) and (3.11.30) improve the estimate (3.11.11). We proceed in the same way to ameliorate the remaining estimates, using (3.11.28) and (3.11.27). Consequently, the solution $(\varphi^{(2)},\widetilde{g}^{(2)})$ exists in $[0,T]$ and we have the following estimate for $\varphi-\varphi^{(2)}$

$$|Z^I(\varphi-\varphi^{(2)})|\le\frac{C\varepsilon^3}{\sqrt{T}\sqrt{1+s}(1+q)^{\frac{1}{2}-2\kappa-5\rho}},\ for\ I\le N-20 \tag{3.11.34}$$

$$\|\alpha_2 w_0^{\frac{1}{2}}\partial Z^I(\varphi-\varphi^{(2)})\|\le\frac{C\varepsilon^3(1+t)^{C\sqrt{\varepsilon}}}{\sqrt{T}},\ for\ I\le N-4. \tag{3.11.35}$$

We now go to the amelioration of the estimate for $\widetilde{b}$. In view of the definition (3.11.1) of $\widetilde{b}^{(2)}$ we have for $I\le N-4$.

$$\begin{aligned}&\partial_\theta^I\left(\widetilde{b}^{(2)}(\theta)-\Pi\int_{\Sigma_{T,\theta}}(\partial_q\varphi^{(2)})^2 r dr\right)\\ =&\partial_\theta^I\Pi\int_{\Sigma_{T,\theta}}\left((\partial_q\varphi)^2-(\partial_q\varphi^{(2)})^2\right)rdr\\ =&\sum_{I_1+I_2\le J}\Pi\int_{\Sigma_{T,\theta}}\partial_\theta^{I_1}(\partial_q\varphi+\partial_q\varphi^{(2)})\partial_\theta^{I_2}(\partial_q\varphi-\partial_q\varphi^{(2)})rdr.\end{aligned}$$

We estimate, for $I_1\le\frac{N}{2}$

$$\begin{aligned}&\left\|\int_{\Sigma_{T,\theta}}\partial_\theta^{I_1}(\partial_q\varphi+\partial_q\varphi^{(2)})\partial_\theta^{I_2}(\partial_q\varphi-\partial_q\varphi^{(2)})rdr\right\|_{L^2(\mathbb{S}^1)}\\ \lesssim&\int_{\Sigma_{T,\theta}}\frac{\varepsilon}{\sqrt{1+s}(1+|q|)^{\frac{3}{2}-4\rho}}\left\|\partial_\theta^{I_2}(\partial_q\varphi-\partial_q\varphi^{(2)})\right\|_{L^2(\mathbb{S}^1)}rdr\\ \lesssim&\left(\int_0^\infty\frac{\varepsilon^2}{(1+s)(1+|q|)^{3-8\rho-4\kappa}}rdr\right)^{\frac{1}{2}}\left\|\beta_2\partial_\theta^{I_2}(\partial_q\varphi-\partial_q\varphi^{(2)})\right\|_{L^2}\end{aligned}$$

Then the estimate (3.11.35), with the condition $(1+T)^{C\sqrt{\varepsilon}}\le 1$ yields for $I\le N-4$

$$\left|\partial_\theta^I\left(\widetilde{b}^{(2)}(\theta)-\Pi\int_{\Sigma_{T,\theta}}(\partial_q\varphi^{(2)})^2 rdr\right)\right|_{L^2(\mathbb{S}^1)}\lesssim\frac{\varepsilon^4}{\sqrt{T}}. \tag{3.11.36}$$

The case $I_2 \leq \frac{N}{2}$ can be treated similarly thanks to (3.11.34). To conclude, we estimate

$$\begin{aligned}
\left\|\partial_\theta^I \widetilde{b}^{(2)}\right\|_{L^2(\mathbb{S}^2)} &= \left\| \int_0^\infty \sum_{I_1+I_2=I} \partial_q \partial_\theta^{I_1} \varphi \partial_q \partial_\theta^{I_2} \varphi r dr \right\|_{L^2(\mathbb{S}^1)} \\
&\leq \int_0^\infty \frac{C_0 \varepsilon}{\sqrt{1+s}(1+|q|)^{\frac{3}{2}-4\rho}} \|\partial_q \partial_\theta^{I_2} \varphi\|_{L^2(\mathbb{S}^1)} r dr \\
&\leq \left(\int \frac{C_0^2 \varepsilon^2}{(1+s)(1+|q|)^{3-8\rho}} r dr \right)^{\frac{1}{2}} \|\partial_q \partial_\theta^{I_2} \varphi\|_{L^2} \\
&\leq 2C_0^2 \varepsilon^2
\end{aligned}$$

where we have used again $(1+T)^{C\sqrt{\varepsilon}} \leq 1$. This concludes the proof of Proposition 3.11.1, and the proof of Theorem 3.1.12.

3.12 Appendix

3.12.1 Construction of the initial data

Theorem 3.1.3 is a consequence of the following result on the constraint equations, proved in Chapter 2. The method of solving is inspired from the conformal method in three dimension. We look for space-like metrics $\bar{g}$ of the form $\bar{g} = e^{2\lambda}\delta$. We introduce the traceless part of K,

$$H_{ij} = K_{ij} - \frac{1}{2}\tau \bar{g}_{ij},$$

and the following rescaling

$$\dot{\varphi} = \frac{e^\lambda}{N}\partial_0 u, \quad \breve{H} = e^{-\lambda}H, \quad \breve{\tau} = e^\lambda \tau.$$

We also introduce the notation

$$M_\theta = \begin{pmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{pmatrix}, \quad N_\theta = \begin{pmatrix} -\sin(2\theta) & \cos(2\theta) \\ \cos(2\theta) & \sin(2\theta) \end{pmatrix}.$$

Theorem 3.12.1. *Let $0 < \delta < 1$. Let $\dot{\varphi}^2, |\nabla\varphi|^2 \in H^{N-1}_{\delta+2}$ and $\bar{b} \in W^{N,2}(\mathbb{S}^1)$ such that*

$$\int_{\mathbb{S}^1} \bar{b}(\theta)\cos(\theta)d\theta = \int_{\mathbb{S}^1} \bar{b}(\theta)\sin(\theta)d\theta = 0.$$

We note

$$\varepsilon^2 = \int \dot{\varphi}^2 + |\nabla\varphi|^2.$$

We assume

$$\|\dot{\varphi}^2\|_{H^{N-1}_{\delta+2}} + \||\nabla\varphi|^2\|_{H^{N-1}_{\delta+2}} + \|\bar{b}\|_{W^{N,2}} \lesssim \varepsilon^2.$$

Let $B \in W^{N,2}(\mathbb{S}^1)$. We assume

$$\|B\|_{W^{N,2}} \lesssim \varepsilon^4.$$

Let $\Psi \in H^{N+1}_{\delta+1}$ be such that $\int \Psi = 2\pi$. If $\varepsilon > 0$ is small enough, there exist $\alpha, \rho, \eta, A, J, c_1, c_2$ in $\mathbb{R}$, a scalar function $\widetilde{\lambda} \in H^{N+1}_\delta$ and a symmetric traceless tensor $\widetilde{H} \in H^N_{\delta+1}$ such that, if r, θ are the polar coordinates centered in c_1, c_2, and if we note

$$\begin{aligned}
\lambda &= -\alpha\chi(r)\ln(r) + \widetilde{\lambda}, \\
\breve{H} &= -(\bar{b}(\theta) + \rho\cos(\theta-\eta))\frac{\chi(r)}{2r}M_\theta + e^{-\lambda}\frac{\chi(r)}{r^2}\left((J-(1-\alpha)B(\theta))N_\theta - \frac{B'(\theta)}{2}M_\theta\right) + \widetilde{H},
\end{aligned}$$

then $\lambda, e^{\lambda}\breve{H}$ are solutions of the constraint equations with

$$\breve{\tau} = (\bar{b}(\theta) + \rho\cos(\theta - \eta))\frac{\chi(r)}{r} + e^{-\lambda}B'(\theta)\frac{\chi(r)}{r^2} + A\psi.$$

Moreover we have the estimates

$$\begin{aligned}
\alpha &= \frac{1}{4\pi}\int \left(\dot{\varphi}^2 + |\nabla\varphi|^2\right) + O(\varepsilon^4),\\
\rho\cos(\eta) &= \frac{1}{\pi}\int \dot{\varphi}\partial_1\varphi + O(\varepsilon^4),\\
\rho\sin(\eta) &= \frac{1}{\pi}\int \dot{\varphi}\partial_2\varphi + O(\varepsilon^4),\\
c_1 &= -\frac{1}{4\pi}\int x_1\left(\dot{\varphi}^2 + |\nabla\varphi|^2\right) + O(\varepsilon^4),\\
c_2 &= -\frac{1}{4\pi}\int x_2\left(\dot{\varphi}^2 + |\nabla\varphi|^2\right) + O(\varepsilon^4),\\
J &= -\frac{1}{2\pi}\int \dot{\varphi}\partial_\theta\varphi + \frac{\rho}{\alpha}(c_2\cos(\eta) - c_1\sin(\eta)) + O(\varepsilon^4),\\
A &= -\frac{1}{2\pi}\int \dot{\varphi}r\partial_r\varphi + \frac{1}{2\pi}\left(\int \chi'(r)rdr\right)\int \bar{b}(\theta)d\theta + O(\varepsilon^4),
\end{aligned}$$

and

$$\|\widetilde{\lambda}\|_{H^{N+1}_\delta} + \|\widetilde{H}\|_{H^N_{\delta+1}} \lesssim \varepsilon^2.$$

We will use the notation

$$b^{(1)} = \rho\cos(\theta - \eta) + \bar{b}(\theta). \tag{3.12.1}$$

The end of this section is devoted to the proof of Theorem 3.1.3.

Lemma 3.12.2. *The second fundamental form of the space-time metric*

$$g_a = -dt^2 - 2Jdtd\theta + r^{-2\alpha}(dr^2 + (r - b^{(1)}(\theta)r^\alpha t)^2 d\theta^2) - 2B'(\theta)td\theta^2 + 4(1-\alpha)B(\theta)\frac{t}{r}drd\theta. \tag{3.12.2}$$

is given at $t = 0$ by

$$K_{ij} = H_{ij} + \frac{1}{2}(g_a)_{ij}\tau,$$

with

$$\tau = r^\alpha\frac{b^{(1)}(\theta)}{r} + r^{2\alpha}\frac{B'(\theta)}{r},$$

$$H = -r^{-\alpha}b^{(1)}(\theta)\frac{\chi(r)}{2r}M_\theta + (J - (1-\alpha)B(\theta))\frac{\chi(r)}{r^2}N_\theta - B'(\theta)\frac{\chi(r)}{2r^2}M_\theta.$$

Proof of Lemma 3.12.2. The metric induced by g_a on the space-like hypersurface $t = 0$ is $r^{-2\alpha}\delta$. The shift is given by $\beta_\theta = -J$ and the lapse is given by $N = 1$. Therefore we calculate

$$K_{ij} = -\frac{1}{2N}(\partial_t\bar{g}_{ij} - \partial_i\beta_j - \partial_j\beta_i).$$

We infer

$$K_{11} = -\frac{1}{2}\left(-\left(\frac{2r^{-\alpha}b^{(1)}(\theta)}{r} + \frac{2B'(\theta)}{r^2}\right)\sin^2(\theta) - \frac{4(1-\alpha)B(\theta)}{r^2}\cos(\theta)\sin(\theta) + \frac{2J}{r^2}\cos(\theta)\sin(\theta)\right),$$

$$K_{22} = -\frac{1}{2}\left(-\left(\frac{2r^{-\alpha}b^{(1)}(\theta)}{r} + \frac{2B'(\theta)}{r^2}\right)\cos^2(\theta) + \frac{4(1-\alpha)B(\theta)}{r^2}\cos(\theta)\sin(\theta) - \frac{2J}{r^2}\cos(\theta)\sin(\theta)\right),$$

$$\begin{aligned} K_{12} = -\frac{1}{2}\Bigg(&\left(\frac{2r^{-\alpha}b^{(1)}(\theta)}{r} + \frac{2B'(\theta)}{r^2}\right)\cos(\theta)\sin(\theta) + \frac{4(1-\alpha)B(\theta)}{2r^2}(\cos^2(\theta) - \sin^2(\theta)) \\ &- \frac{2J}{r^2}(\cos^2(\theta) - \sin^2(\theta))\Bigg). \end{aligned}$$

We calculate

$$\tau = \bar{g}^{ij}K_{ij} = r^{\alpha}\frac{b^{(1)}(\theta)}{r} + r^{2\alpha}\frac{B'(\theta)}{r^2}$$

so we obtain exactly

$$H = -r^{-\alpha}b^{(1)}(\theta)\frac{\chi(r)}{2r}M_\theta + (J - (1-\alpha)B(\theta))\frac{\chi(r)}{r^2}N_\theta - B'(\theta)\frac{\chi(r)}{2r^2}M_\theta.$$

□

Lemma 3.12.3. *The metric g_a, defined by* (3.12.2) *is isometric to $g_b + g^{(1)}$ where at $t = 0$ we have*

$$g^{(1)} = O\left(\frac{1}{r^2}\right), \quad \partial_t g^{(1)} = O\left(\frac{1}{r^2}\right),$$

and g_b is defined by (3.1.6)*, where*

$$b(\theta) = \frac{b^{(1)}(F(\theta))}{1-\alpha-b^{(1)}(F(\theta))}, \tag{3.12.3}$$

$$J(\theta) = 2JF'(\theta), \tag{3.12.4}$$

with F the inverse function of

$$\theta \mapsto \theta + \int_0^\theta (\alpha - b^{(1)}(\theta')d\theta';$$

provided the following relations hold

$$\alpha = -\int \bar{b}(\theta), \tag{3.12.5}$$

$$B(\theta) = \frac{Jb^{(1)}(\theta)}{1-\alpha}. \tag{3.12.6}$$

Proof. During all the proof, the notation $g \sim g'$ stands for g is isometric to $g' + \widetilde{g}$ where $\widetilde{g} = O\left(\frac{1}{r}\right)$ and $\partial_t \widetilde{g} = O\left(\frac{1}{r^3}\right)$. In polar coordinate r, θ, this means neglecting the metric terms of the form

$$\frac{dr^2}{r^2}, \quad \frac{d\theta dr}{r}, \quad d\theta^2,$$

$$\frac{tdr^2}{r^3}, \quad \frac{td\theta dr}{r^2}, \quad \frac{td\theta^2}{r}.$$

We perform some changes of variable in g_{as}. First of all we introduce r' such that

$$r' = \frac{r^{1-\alpha}}{1-\alpha}, \quad dr' = r^{-\alpha}dr.$$

The metric g_a becomes

$$g_a \sim -dt^2 - 2Jdtd\theta + (dr)^2 + (r(1-\alpha) - b^{(1)}(\theta)t)^2 d\theta^2 - 2B'(\theta)td\theta^2 + 4B(\theta)\frac{t}{r}drd\theta,$$

where we keep writing r instead of r'. We now make the change of variable

$$\theta = \theta' - \frac{J}{(1-\alpha)^2 r}, \quad d\theta = d\theta' + \frac{J}{(1-\alpha)^2 r^2}.$$

Since we will neglect the contributions to the metric decaying like $\frac{1}{r^2}$ we obtain

$$d\theta^2 \sim (d\theta')^2 + 2\frac{J}{(1-\alpha)^2 r^2}d\theta' dr, \quad b^{(1)}(\theta) \sim b(\theta') - b'(\theta')\frac{J}{r(1-\alpha)^2}.$$

We keep also writing θ instead of θ'. We infer

$$\begin{aligned} g_a \sim &- dt^2 - 2J(dt-dr)d\theta + dr^2 + (r(1-\alpha) - b^{(1)}(\theta)t)^2 d\theta^2 \\ &+ \left(2\frac{Jb'(\theta)}{(1-\alpha)} - 2B'(\theta)\right)td\theta^2 + \left(-4b^{(1)}(\theta)\frac{J}{1-\alpha} + 4B(\theta)\right)\frac{t}{r}drd\theta. \end{aligned}$$

We choose

$$B(\theta) = \frac{Jb^{(1)}(\theta)}{1-\alpha}.$$

With this choice we obtain

$$\begin{aligned} g_a &\sim - dt^2 - 2J(dt-dr)d\theta + dr^2 + (r(1-\alpha) - b^{(1)}(\theta)t)^2 d\theta^2 \\ &\sim - dt^2 - 2J(dt-dr)d\theta + dr^2 + (r - (b^{(1)}(\theta)+\alpha)r + b^{(1)}(\theta)(r-t))^2 d\theta^2. \end{aligned}$$

We impose

$$\alpha = -\int b^{(1)}(\theta) = -\int \bar{b}(\theta).$$

Therefore we can find $f(\theta)$ such that

$$f'(\theta) = -(b^{(1)}(\theta) + \alpha).$$

We perform the change of variable

$$\theta' = \theta + f(\theta).$$

We note F the inverse function of

$$\theta \mapsto \theta + f(\theta),$$

so that $\theta = F(\theta')$. Then g_a becomes

$$g_a \sim -dt^2 - 2JF'(\theta')(dt-dr)d\theta' + dr^2 + \left(r + \frac{b^{(1)}(F(\theta'))}{1-\alpha-b^{(1)}(F(\theta'))}(r-t)\right)^2 d(\theta')^2.$$

We set

$$b(\theta') = \frac{b^{(1)}(F(\theta'))}{1-\alpha-b^{(1)}(F(\theta'))},$$
$$J(\theta') = 2JF'(\theta').$$

Let us note that J is at the same level of regularity than b. $\square$

We are now ready to prove Theorem 3.1.3.

Proof of Theorem 3.1.3. We consider the map

$$\Phi : \bar{b} \mapsto \Pi b,$$

where

- $\bar{b} \in W^{N,2}$ is such that
$$\int \bar{b}\cos(\theta) = \int \bar{b}\sin(\theta) = 0, \quad \alpha = -\int \bar{b}(\theta),$$
where α is given by Theorem 3.12.1,
- b is given by formula (3.12.3), where $b^{(1)} = \rho\cos(\theta - \eta) + \bar{b}$, and ρ, η are given by Theorem 3.12.1.
- Π is the projection
$$\Pi : W^{2,N}(\mathbb{S}^1) \to \{u \in W^{2,N}(\mathbb{S}^1), \int u = \int \cos(\theta)u = \int \sin(\theta)u = 0\}. \tag{3.12.7}$$

It is easy to see that Φ is invertible for ε small enough. Therefore, for $\widetilde{b} \in W^{2,N}$ such that

$$\int \widetilde{b} = \int \widetilde{b}\cos(\theta) = \int \widetilde{b}\sin(\theta) = 0,$$

we apply Theorem 3.12.1 to $\Phi^{-1}(\widetilde{b})$. Thanks to Lemma 3.12.2 and 3.12.3 we can find $(g_0)_{ij} \in H^{N+1}_\delta$ and $(K_0)_{ij} \in H^N_{\delta+1}$ such that $(g_b)_{ij} + (g_0)_{ij}$ and $(K_b)_{ij} + (K_0)_{ij}$ satisfy the constraint equations, where we have noted K_b the second fundamental form associated to g_b. We complete the initial data as follow. We write our metric in the form $g = g_b + \widetilde{g}$. The initial data for $\widetilde{g}$ are the following

- $\widetilde{g}_{ij}$ is given by $\widetilde{g}_{ij} = (\widetilde{g}_0)_{ij}$,
- $\widetilde{g}_{00}$ and $\widetilde{g}_{0i}$ are taken to be 0[1]
- $\partial_t\widetilde{g}_{ij}$ is given by the relation $\partial_0 g_{ij} = -2NK_{ij}$ and $K_{ij} = (K_b)_{ij} + (K_0)_{ij}$.
- $\partial_t\widetilde{g}_{00}$ and $\partial_t\widetilde{g}_{0i}$ are chosen such that the generalized wave coordinate condition is satisfied at $t = 0$.

Let us describe the last point. The generalized wave coordinate condition writes

$$g^{\lambda\beta}\Gamma^\alpha_{\lambda\beta} = H^\alpha_b = (g_b)^{\lambda\beta}(\Gamma_b)^\alpha_{\lambda\beta} + F^\alpha,$$

Therefore, if we write it for $\alpha = i$ we obtain a relation for $\partial_t g_{0i}$ and if we write it for $\alpha = 0$, we obtain a relation for $\partial_t g_{00}$. However, if we write $g = g_b + \widetilde{g}$, the term

$$g^{\lambda\beta}\Gamma^\alpha_{\lambda\beta} - (g_b)^{\lambda\beta}(\Gamma_b)^\alpha_{\lambda\beta}$$

contains crossed terms of the form

$$\widetilde{g}\partial_U g_b \sim \widetilde{g}\frac{\partial_\theta b(\theta)}{r}.$$

which do not belong in $H^N_{\delta+1}$ because we are missing a derivative on b, since $b \in W^{2,N}$. Therefore, we will take F^α as defined in (3.1.9). With this choice, the generalized wave coordinate condition imply that $\partial_t\widetilde{g}_{00}$ and $\partial_t\widetilde{g}_{0i}$ are given by a sum of terms the form

$$K_0,\ \nabla g_0,\ g_b K_0,\ g_b\nabla g_0,\ \frac{\chi(r)g_b}{r}g_0.$$

With this choice, $\partial_t\widetilde{g}_{0i}$ and $\partial_t\widetilde{g}_{00}$ belong to $H^N_{\delta+1}$. □

[1] The lapse ans shift are given by g_b: we have $N = 1$ and $\beta_r = 0$ and $\beta_\theta = -J$.

3.12.2 The generalised wave coordinates

In a coordinate system, the Ricci tensor is given by

$$R_{\mu\nu} = \partial_\alpha \Gamma^\alpha_{\mu\nu} - \partial_\mu \Gamma^\alpha_{\alpha\nu} + \Gamma^\alpha_{\mu\nu}\Gamma^\lambda_{\alpha\lambda} - \Gamma^\alpha_{\mu\lambda}\Gamma^\lambda_{\nu\alpha}, \tag{3.12.8}$$

where the $\Gamma^\lambda_{\alpha\beta}$ are the Christoffel symbols given by

$$\Gamma^\lambda_{\alpha\beta} = \frac{1}{2} g^{\lambda\rho}\left(\partial_\alpha g_{\rho\beta} + \partial_\beta g_{\rho\alpha} - \partial_\rho g_{\alpha\beta}\right). \tag{3.12.9}$$

$R_{\mu\nu}$ an operator of order two for g. In order to single out the hyperbolic part, we will write

$$H^\alpha = g^{\lambda\beta}\Gamma^\alpha_{\lambda\beta}, \tag{3.12.10}$$

which can also be written

$$H^\alpha = -\partial_\lambda g^{\lambda\alpha} - \frac{1}{2} g^{\lambda\mu}\partial^\alpha g_{\lambda\mu}.$$

We compute $R_{\mu\nu}$ in terms of g and H.

$$\begin{aligned}
R_{\mu\nu} =& \frac{1}{2}\partial_\alpha \left(g^{\alpha\rho}(\partial_\mu g_{\rho\nu} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu})\right) - \frac{1}{2}\partial_\mu\left(g^{\alpha\rho}(\partial_\nu g_{\rho\alpha} + \partial_\alpha g_{\rho\nu} - \partial_\rho g_{\nu\alpha})\right)\\
&+ \frac{1}{4} g^{\alpha\rho} g^{\lambda\beta}(\partial_\mu g_{\rho\nu} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu})(\partial_\lambda g_{\beta\alpha} + \partial_\alpha g_{\beta\lambda} - \partial_\beta g_{\alpha\lambda})\\
&- \frac{1}{4} g^{\alpha\rho} g^{\lambda\beta}(\partial_\nu g_{\rho\lambda} + \partial_\lambda g_{\rho\nu} - \partial_\rho g_{\nu\lambda})(\partial_\mu g_{\alpha\beta} + \partial_\alpha g_{\beta\mu} - \partial_\beta g_{\alpha\mu}),
\end{aligned}$$

$$R_{\mu\nu} = -\frac{1}{2} g^{\alpha\rho}\partial_\alpha\partial_\rho g_{\mu\nu} + \frac{1}{2} H^\rho \partial_\rho g_{\mu\nu} + \frac{1}{2}\left(g_{\mu\rho}\partial_\nu H^\rho + g_{\nu\rho}\partial_\mu H^\rho\right) + \frac{1}{2} P_{\mu\nu}(g)(\partial g, \partial g), \tag{3.12.11}$$

with

$$\begin{aligned}
P_{\mu\nu}(g)(\partial g, \partial g) =& \frac{1}{2} g^{\alpha\rho} g^{\beta\sigma}\left(\partial_\mu g_{\rho\sigma}\partial_\alpha g_{\beta\nu} + \partial_\nu g_{\rho\sigma}\partial_\alpha g_{\beta\mu} - \partial_\beta g_{\mu\rho}\partial_\alpha g_{\nu\sigma} - \frac{1}{2}\partial_\mu g_{\alpha\beta}\partial_\nu g_{\rho\sigma}\right)\\
&+ \frac{1}{2} g^{\alpha\beta} g^{\lambda\rho}\partial_\alpha g_{\nu\rho}\partial_\beta g_{\mu\rho}
\end{aligned} \tag{3.12.12}$$

Proposition 3.12.4. *If the coupled system of equations*

$$\begin{cases} -\frac{1}{2} g^{\alpha\rho}\partial_\alpha\partial_\rho g_{\mu\nu} + \frac{1}{2} F^\rho\partial_\rho g_{\mu\nu} + \frac{1}{2}\left(g_{\mu\rho}\partial_\nu F^\rho + g_{\nu\rho}\partial_\mu F^\rho\right) + \frac{1}{2} P_{\mu\nu}(g)(\partial g, \partial g) = \partial_\mu\varphi\partial_\nu\varphi \\ g^{\alpha\rho}\partial_\alpha\partial_\rho\varphi - F^\rho\partial_\rho\varphi = 0 \end{cases}$$

with F a function which may depend on φ, g, is satisfied on a time interval $[0,T]$ with $T > 0$, if the initial induced Riemannian metric and second fundamental form $(\bar{g}, K)$ satisfy the constraint equations, and if the initial compatibility condition

$$F^\alpha|_{t=0} = H^\alpha|_{t=0}, \tag{3.12.13}$$

is satisfied, then for all time, the equations (3.1.1) *are satisfied on* $[0,T]$*, together with the wave coordinate condition*

$$F^\alpha = H^\alpha.$$

Proof. We use the twice contracted Bianchi Identity

$$D^\mu\left(R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}\right) = 0.$$

with H defined by (3.12.10). Since in $[0,T]$, we have

$$-\frac{1}{2}g^{\alpha\rho}\partial_\alpha\partial_\rho g_{\mu\nu}+\frac{1}{2}F^\rho\partial_\rho g_{\mu\nu}+\frac{1}{2}\left(g_{\mu\rho}\partial_\nu F^\rho+g_{\nu\rho}\partial_\mu F^\rho\right)+P_{\mu\nu}(g)(\partial g,\partial g)=\partial_\mu\varphi\partial_\nu\varphi.$$

Thanks to (3.12.11) we obtain

$$\frac{1}{2}(F^\rho-H^\rho)\partial_\rho g_{\mu\nu}+\frac{1}{2}\left(g_{\mu\rho}\partial_\nu(F^\rho-H^\rho)+g_{\nu\rho}\partial_\mu(F^\rho-H^\rho)\right)=\partial_\mu\varphi\partial_\nu\varphi-R_{\mu\nu}.$$

Consequently, since $D^\mu\left(R_{\mu\nu}-\frac{1}{2}Rg_{\mu\nu}\right)=0$ and $D^\mu\left(\partial_\mu\varphi\partial_\nu\varphi-\frac{1}{2}g_{\mu\nu}\partial^\alpha\varphi\partial_\alpha\varphi\right)=0$ and we obtain the following equation on $F^\rho-H^\rho$

$$\begin{aligned}0=&D^\mu\Bigg(\frac{1}{2}\left(g_{\mu\rho}\partial_\nu(F^\rho-H^\rho)+g_{\nu\rho}\partial_\mu(F^\rho-H^\rho)\right)-\frac{1}{4}g_{\mu\nu}g^{\alpha\beta}\left(g_{\alpha\rho}\partial_\beta(F^\rho-H^\rho)+g_{\alpha\rho}\partial_\beta(F^\rho-H^\rho)\right)\\&+\frac{1}{2}\left(\partial_\rho g_{\mu\nu}-\frac{1}{2}g^{\alpha\beta}\partial_\rho g_{\alpha\beta}\right)(F^\alpha-H^\alpha)\Bigg)\end{aligned}$$

Multiplying by $g^{\nu\alpha}$ we obtain

$$\Box_g(F^\alpha-H^\alpha)+B^{\alpha,\beta}_\rho\partial_\beta(F^\rho-H^\rho)+C^\alpha_\rho(F^\rho-H^\rho)=0,$$

with $B^{\alpha,\beta}_\rho$, C^α_ρ coefficients depending on g,φ, well defined in $[0,T]$. This is an equation in hyperbolic form, therefore if the initial data $(F^\alpha-H^\alpha)|_{t=0}$ and $\partial_t(F^\alpha-H^\alpha)|_{t=0}$ are zero, then the solution is identically zero on $[0,T]$. Since we assume (3.12.13), we only have to check

$$\partial_t(F^\alpha-H^\alpha)|_{t=0}=0.$$

Since the constraint equations are satisfied, we have

$$\begin{aligned}R_{0i}&=\partial_0\varphi\partial_i\varphi,\\R_{00}-\frac{1}{2}g_{00}R&=\partial_0\varphi\partial_0\varphi-\frac{1}{2}g_{00}\partial^\mu\varphi\partial_\mu\varphi.\end{aligned}$$

Therefore, using once again equation (3.12.11) and (3.12.13) we obtain

$$\begin{aligned}0=&g_{i\rho}\partial_t(F^\rho-H^\rho),\\0=&2g_{0\rho}\partial_t(F^\rho-H^\rho)-g_{00}\partial_t(F^0-H^0).\end{aligned}$$

This system can be written as

$$\begin{pmatrix}g_{00}&2g_{01}&2g_{02}\\g_{01}&g_{11}&g_{12}\\g_{02}&g_{12}&g_{22}\end{pmatrix}\begin{pmatrix}\partial_t(F^0-H^0)\\\partial_t(F^1-H^1)\\\partial_t(F^2-H^2)\end{pmatrix}=0.$$

It is invertible so $\partial_t(F^\rho-H^\rho)_{t=0}=0$. Therefore in $[0,T]$ we have $F^\rho=H^\rho$ and equation (3.12.11) implies that the Einstein Equations (3.1.1) are satisfied. $\Box$

3.12.3 The $L^\infty-L^\infty$ estimate

For the sake of completeness, we give here the proof of the $L^\infty-L^\infty$ estimate by Kubo and Kubota (see [35]).

Proposition 3.12.5. *Let u be a solution of*

$$\begin{cases}\Box u=F,\\(u,\partial_t u)|_{t=0}=(0,0),\end{cases}$$

The $L^\infty - L^\infty$ estimate writes: for $\mu > \frac{3}{2}, \nu > 1$

$$|u(t,x)|(1+t+|x|)^{\frac{1}{2}} \le C(\mu,\nu) M_{\mu,\nu}(f)(1+|t-|x||)^{-\frac{1}{2}+[2-\mu]_+},$$

where

$$M_{\mu,\nu}(f) = \sup(1+|y|+s)^\mu (1+|s-|y||)^\nu |F(y,s)|,$$

and we have the convention $A^{[0]_+} = \ln(A)$.

Proof. We write the solution u of

$$\begin{cases} \Box u = F, \\ (u, \partial_t u)|_{t=0} = (0,0), \end{cases}$$

with the representation formula

$$u(x,t) = \int_0^t \int_{|y|\le t-s} \frac{1}{\sqrt{(t-s)^2 - |y|^2}} F(s, x-y) dyds.$$

With $M_{\mu,\nu}(f) = \sup(1+|y|+s)^\mu (1+|s-|y||)^\nu |F(y,s)|$, we can write

$$|u(x,t)| \le M_{\mu,\nu}(f) \int_0^t \int_{|y|\le t-s} \frac{1}{\sqrt{(t-s)^2-|y|^2}} \frac{1}{(1+|x-y|+s)^\mu (1+|s-|x-y||)^\nu} dyds.$$

It is therefore sufficient to study the quantity

$$I(x,t) = \int_0^t \int_{|y|\le t-s} \frac{1}{\sqrt{(t-s)^2-|y|^2}} \frac{1}{(1+|x-y|+s)^\mu (1+|s-|x-y||)^\nu} dyds.$$

We begin with a lemma on spherical means.

Lemma 3.12.6. *Let $b \in \mathcal{C}^0(\mathbb{R}^2)$. We have the following equality for $\rho \ge 0$*

$$\int_{|\omega|=1} b(|x+\rho\omega|) d\omega = 4 \int_{|\rho-r|}^{\rho+r} \lambda b(\lambda) h(\lambda,\rho,r) d\lambda,$$

where we note $r = |x|$ and

$$\begin{aligned} h(\lambda,\rho,r) &= \left(\lambda^2 - (\rho-r)^2\right)^{-\frac{1}{2}} \left((\rho+r)^2 - \lambda^2\right)^{-\frac{1}{2}} \\ &= \left((\lambda+r)^2 - \rho^2\right)^{-\frac{1}{2}} \left(\rho^2 - (\lambda-r)^2\right)^{-\frac{1}{2}}. \end{aligned}$$

Proof. By eventually rotating the axis, we can assume $x = (r,0)$ in (x_1,x_2) coordinates. Therefore we have

$$\int_{|\omega|=1} b(|x+\rho\omega|) d\omega = \int_0^{2\pi} b\left((r^2+\rho^2+2r\rho\cos(\theta))^{\frac{1}{2}}\right) d\theta = 2\int_0^\pi b\left((r^2+\rho^2+2r\rho\cos(\theta))^{\frac{1}{2}}\right) d\theta.$$

We make the change of variable $\lambda = (r^2+\rho^2+2\rho r\cos(\theta))^{\frac{1}{2}}$, for $\theta \in [0,\pi[$. Then we have

$$\begin{aligned} d\lambda &= -\frac{1}{\lambda}\rho r \sin(\theta) d\theta \\ &= -\frac{1}{\lambda}\rho r \left(1 - \frac{(\lambda^2-r^2-\rho^2)^2}{(2\rho r)^2}\right)^{\frac{1}{2}} d\theta \\ &= -\frac{1}{2\lambda}\left((2\rho r)^2 - (\lambda^2-r^2-\rho^2)^2\right)^{\frac{1}{2}} d\theta \\ &= -\frac{1}{2\lambda}\left(2\rho r - \lambda^2 + r^2 + \rho^2\right)^{\frac{1}{2}} \left(2\rho r + \lambda^2 - \rho^2 - r^2\right)^{\frac{1}{2}} d\theta. \end{aligned}$$

We have therefore $d\theta = -2\lambda h(\lambda,\rho,r) d\lambda$, which concludes the proof of Lemma 3.12.6. □

We use Lemma 3.12.6 to calculate I

$$\begin{aligned}
I(x,t) &= \int_0^t \int_{\rho\le t-s} \frac{\rho}{\sqrt{(t-s)^2-\rho^2}} \int_{|\omega|=1} \frac{1}{(1+|x+\rho\omega|+s)^\mu(1+|s-|x+\rho\omega||)^\nu} d\omega d\rho ds \\
&= 4\int_0^t \int_{\rho\le t-s} \frac{\rho}{\sqrt{(t-s)^2-\rho^2}} \int_{|\rho-r|}^{\rho+r} \frac{h(\lambda,\rho,r)}{(1+\lambda+s)^\mu(1+|s-\lambda|)^\nu} \lambda d\lambda d\rho ds.
\end{aligned}$$

We exchange the integration in ρ with the integration in λ, noticing that

$$1\!\!1_{|\rho-r|\le\lambda\le\rho+r} = 1\!\!1_{|\lambda-r|\le\rho\le\lambda+r},$$

and we make the decomposition $I = I_1 + I_2$, separating the region $\lambda + r \le t - s$ from $\lambda + r \ge t - s$.

$$\begin{aligned}
I_1 &= \int_0^{t-r} \int_{\lambda=0}^{t-s-r} \frac{\lambda}{z(s,\lambda)} \int_{|\lambda-r|}^{\lambda+r} \frac{h(\lambda,\rho,r)}{\sqrt{(t-s)^2-\rho^2}} \rho d\rho d\lambda ds, \\
I_2 &= \int_0^{t} \int_{\lambda=\max(t-s-r,0)}^{t-s+r} \frac{\lambda}{z(s,\lambda)} \int_{|\lambda-r|}^{t-s} \frac{h(\lambda,\rho,r)}{\sqrt{(t-s)^2-\rho^2}} \rho d\rho d\lambda ds,
\end{aligned}$$

where $z(s,\lambda) = (1+\lambda+s)^\mu(1+|s-\lambda|)^\nu$.

3.12.3.1 Estimate of I_1

We write

$$\begin{aligned}
\int_{|\lambda-r|}^{\lambda+r} \frac{h(\lambda,\rho,r)}{\sqrt{(t-s)^2-\rho^2}} \rho d\rho &= \int_{|\lambda-r|}^{\lambda+r} \frac{1}{\sqrt{(t-s)^2-\rho^2}\sqrt{(\lambda+r)^2-\rho^2}\sqrt{\rho^2-(\lambda-r)^2}} \rho d\rho \\
&= \frac{1}{2}\int_a^b \frac{du}{\sqrt{d-u}\sqrt{b-u}\sqrt{u-a}},
\end{aligned}$$

with $a = (\lambda-r)^2$, $b = (\lambda+r)^2$ and $d = (t-s)^2$. Recall that in the integration region of I_1, we have $\lambda + r \le t - s$ so $b \le d$. This yields

$$\int_a^b \frac{du}{\sqrt{d-u}\sqrt{b-u}\sqrt{u-a}} \le \frac{1}{\sqrt{d-b}}\int_a^b \frac{du}{\sqrt{b-u}\sqrt{u-a}} \le \frac{1}{\sqrt{d-b}}\int_0^1 \frac{dv}{\sqrt{v}\sqrt{1-v}} \le \frac{\pi}{\sqrt{d-b}}. \tag{3.12.14}$$

Consequently we have

$$I_1 \lesssim \int_0^{t-r}\int_0^{t-s-r} \frac{\lambda}{\sqrt{(t-s)^2-(\lambda+r)^2}(1+\lambda+s)^\mu(1+|s-\lambda|)^\nu} d\lambda ds.$$

We make the change of variable $\alpha = s - \lambda$, $\beta = \lambda + s$. We obtain

$$I_1 \lesssim \left(\int_0^{t-r} \frac{\beta d\beta}{\sqrt{t-r-\beta}(1+\beta)^\mu}\right)\left(\int_{r-t}^{t-r} \frac{d\alpha}{\sqrt{t+r-\alpha}(1+|\alpha|)^\nu}\right).$$

We estimate the first factor. We note that if $t-r\le 1$, this factor is bounded. We assume therefore that $t - r \ge 1$.

$$\begin{aligned}
\int_0^{t-r} \frac{\beta d\beta}{\sqrt{t-r-\beta}(1+\beta)^\mu} &= \int_0^{\frac{t-r}{2}} \frac{\beta d\beta}{\sqrt{t-r-\beta}(1+\beta)^\mu} + \int_{\frac{t-r}{2}}^{t-r} \frac{\beta d\beta}{\sqrt{t-r-\beta}(1+\beta)^\mu} \\
&\lesssim \frac{1}{\sqrt{t-r}}\int_0^{\frac{t-r}{2}} \frac{\beta d\beta}{(1+\beta)^\mu} + (t-r)^{1-\mu}\int_{\frac{t-r}{2}}^{t-r} \frac{d\beta}{\sqrt{t-r-\beta}} \\
&\lesssim \frac{(t-r)^{[2-\mu]_+}}{\sqrt{t-r}}.
\end{aligned}$$

We estimate the second factor

$$\begin{aligned}\int_{r-t}^{t-r} \frac{d\alpha}{\sqrt{t+r-\alpha}(1+|\alpha|)^{\nu}} &= \int_{r-t}^{\min(\frac{t+r}{2},t-r)} \frac{d\alpha}{\sqrt{t+r-\alpha}(1+|\alpha|)^{\nu}} + \int_{\min(\frac{t+r}{2},t-r)}^{t-r} \frac{d\alpha}{\sqrt{t+r-\alpha}(1+|\alpha|)^{\nu}} \\ &\lesssim \frac{1}{\sqrt{t+r}} \int_{r-t}^{\min(\frac{t+r}{2},t-r)} \frac{d\alpha}{(1+|\alpha|)^{\nu}} + \frac{1}{(1+t+r)^{\nu}} \int_{\min(\frac{t+r}{2},t-r)}^{t-r} \frac{d\alpha}{\sqrt{t+r-\alpha}} \\ &\lesssim \frac{1}{\sqrt{t+r}},\end{aligned}$$

where we have used in the last inequality the fact that $\nu > 1$. We have proved

$$I_1 \lesssim \frac{(1+|t-r|)^{[2-\mu]_+}}{\sqrt{1+t+r}\sqrt{|t-r|}}.$$

3.12.3.2 Estimate of I_2

As in the estimate of I_1, we write

$$\int_{|\lambda-r|}^{t-s} \frac{h(\lambda,\rho,r)}{\sqrt{(t-s)^2-\rho^2}}\rho d\rho = \frac{1}{2}\int_a^d \frac{du}{\sqrt{d-u}\sqrt{b-u}\sqrt{u-a}},$$

with $a = (\lambda - r)^2$, $b = (\lambda + r)^2$ and $d = (t-s)^2$. In the region $\lambda + r \geq t - s$, we have $b \geq d$, therefore as for (3.12.14) we get

$$\frac{1}{2}\int_a^d \frac{du}{\sqrt{d-u}\sqrt{b-u}\sqrt{u-a}} \lesssim \frac{1}{\sqrt{b-d}}\int_a^d \frac{du}{\sqrt{d-u}\sqrt{u-a}}$$

and so

$$\int_{|\lambda-r|}^{t-s} \frac{h(\lambda,\rho,r)}{\sqrt{(t-s)^2-\rho^2}}\rho d\rho \lesssim \frac{1}{\sqrt{(\lambda+r)^2-(t-s)^2}}.$$

Therefore we have

$$I_2 \lesssim \int_0^t \int_{\lambda=\max(t-s-r,0)}^{t-s+r} \frac{\lambda}{\sqrt{(\lambda+r)^2-(t-s)^2}(1+\lambda+s)^{\mu}(1+|s-\lambda|)^{\nu}} d\lambda ds.$$

We make the same change of variable $\alpha = s - \lambda$, $\beta = \lambda + s$. We obtain

$$I_2 \lesssim \left(\int_{\max(0,t-r)}^{t+r} \frac{\beta d\beta}{\sqrt{\beta-(t-r)}(1+\beta)^{\mu}}\right)\left(\int_{-r-t}^{t} \frac{d\alpha}{\sqrt{t+r-\alpha}(1+|\alpha|)^{\nu}}\right).$$

We estimate the first factor. We first assume $t - r > 0$.

$$\begin{aligned}\int_{t-r}^{t+r} \frac{\beta d\beta}{\sqrt{\beta-(t-r)}(1+\beta)^{\mu}} &\lesssim \int_0^{2r} \frac{(\rho+1+t-r)^{1-\mu}}{\sqrt{\rho}} d\rho \\ &\lesssim (1+|t-r|)^{\frac{3}{2}-\mu} \int_0^{\frac{2r}{1+t-r}} \frac{(1+u)^{1-\mu}}{\sqrt{u}} du \\ &\lesssim (1+|t-r|)^{\frac{3}{2}-\mu},\end{aligned}$$

where we have made consecutively the changes of variable $\rho = \beta - |t-r|$ and $u = \frac{\rho}{1+|t-r|}$, and where we use in the last inequality the fact that $\frac{(1+u)^{1-\mu}}{\sqrt{u}}$ is integrable.

We now assume $t-r<-1$. Then

$$\begin{aligned}\int_0^{t+r}\frac{\beta d\beta}{\sqrt{\beta+|t-r|}(1+\beta)^\mu} &\lesssim |t-r|^{\frac{3}{2}}\int_0^{\frac{t+r}{|t-r|}}\frac{\rho}{\sqrt{1+\rho}(1+|t-r|\rho)^\mu}d\rho\\ &\lesssim (1+|t-r|)^{\frac{3}{2}-\mu}\int_0^{\frac{t+r}{|t-r|}}\frac{\rho}{\sqrt{\rho}\left(\frac{1}{|t-r|}+\rho\right)^\mu}d\rho\\ &\lesssim (1+|t-r|)^{-\frac{1}{2}+[2-\mu]_+},\end{aligned}$$

where we have made the change of variable $\rho=\frac{\beta}{|t-r|}$, and also used the fact that $\mu>\frac{3}{2}$.

We estimate the second factor

$$\begin{aligned}\int_{-r-t}^{t}\frac{d\alpha}{\sqrt{t+r-\alpha}(1+|\alpha|)^\nu} &\lesssim \int_{-r-t}^{\min(t,\frac{t+r}{2})}\frac{d\alpha}{\sqrt{t+r-\alpha}(1+|\alpha|)^\nu}+\int_{\min(t,\frac{t+r}{2})}^{t}\frac{d\alpha}{\sqrt{t+r-\alpha}(1+|\alpha|)^\nu}\\ &\lesssim \frac{1}{\sqrt{t+r}}\int_{-r-t}^{\min(t,\frac{t+r}{2})}\frac{d\alpha}{(1+|\alpha|)^\nu}+\frac{1}{(1+t+r)^\nu}\int_{\min(t,\frac{t+r}{2})}^{t}\frac{d\alpha}{\sqrt{t+r-\alpha}}\\ &\lesssim \frac{1}{\sqrt{t+r}},\end{aligned}$$

where we have used the fact that $\nu>1$. We have proved therefore that

$$I_2\lesssim\frac{(1+|t-r|)^{-\frac{1}{2}+[2-\mu]_+}}{\sqrt{1+t+r}},$$

so

$$I\le I_1+I_2\lesssim\frac{(1+|t-r|)^{[2-\mu]_+}}{\sqrt{1+t+r}\sqrt{1+|t-r|}}$$

The proof of the $L^\infty-L^\infty$ estimate is now complete. □

3.12.4 Hardy inequality with weight

Proposition 3.12.7. *Let $\alpha<1$ and $\beta>1$. We have, with $q=r-t$,*

$$\int u^2 f(q)rdrd\theta\le C(\alpha,\rho)\int(\partial_r u)^2 g(q)rdrd\theta$$

where

$$\begin{aligned}f(q)&=(1+|q|)^{\beta-2},\quad q>0\\ &=(1+|q|)^{\alpha-2},\quad q<0\end{aligned}$$

$$\begin{aligned}g(q)&=(1+|q|)^{\beta},\quad q>0\\ &=(1+|q|)^{\alpha},\quad q<0\end{aligned}$$

Proof. We look first at the region $r>t$. We can assume, by a density argument that u is compactly supported. We calculate

$$\partial_r\left(r(1+r-t)^{\beta-1}\right)=(1+r-t)^{\beta-1}+(\beta-1)r(1+r-t)^{\beta-2}=r(1+r-t)^{\beta-2}\left(\frac{1+r-t}{r}+\beta-1\right)$$

We want to find $c>0$ such that

$$\frac{1+r-t}{r}+\beta-1>c.$$

This condition is satisfied if

$$t < 1 + r(\beta - c)$$

which is the case if $\beta - c > 1$. Since $\beta > 1$ we can find such a $c > 0$. Therefore

$$\begin{aligned}
&\int_t^\infty \int_0^{2\pi} u^2(1+r-t)^{\beta-2} r dr d\theta \\
\leq &\frac{1}{c} \int_t^\infty \int_0^{2\pi} u^2 \partial_r \left(r(1+r-t)^{\beta-1} \right) dr d\theta \\
\leq &\frac{1}{c} \left(-\int_t^\infty \int_0^{2\pi} (\partial_r u^2)(1+r-t)^{\beta-1} r dr d\theta + \left[\int_0^{2\pi} u^2(r,\theta)(1+r-t)^{\beta-1} r d\theta \right]_t^\infty \right).
\end{aligned}$$

Since u is compactly supported,

$$\left[\int_0^{2\pi} u^2(r,\theta)(1+r-t)^{\beta-1} r d\theta \right]_t^\infty \leq 0$$

therefore

$$\begin{aligned}
&\int_t^\infty \int_0^{2\pi} u^2(1+r-t)^{\beta-2} r dr d\theta \\
\leq &\frac{2}{c} \int_t^\infty \int_0^{2\pi} |u \partial_r u| (1+r-t)^{\beta-1} r dr d\theta \\
\leq &\frac{2}{c} \left(\int_t^\infty \int_0^{2\pi} u^2(1+r-t)^{\beta-2} r dr d\theta \right)^{\frac{1}{2}} \left(\int_t^\infty \int_0^{2\pi} (\partial_r u)^2 (1+r-t)^{\beta} r dr d\theta \right)^{\frac{1}{2}}
\end{aligned}$$

We have proved

$$\int_t^\infty \int_0^{2\pi} u^2(1+r-t)^{\beta-2} r dr d\theta \leq C(\alpha) \int_t^\infty \int_0^{2\pi} (\partial_r u)^2 (1+r-t)^{\beta} r dr d\theta. \tag{3.12.15}$$

We now look at the region $r < t$. We calculate

$$\partial_r \left(r(1+t-r)^{\alpha-1} \right) = (1+t-r)^{\alpha-1} + (1-\alpha) r (1+t-r)^{\alpha-2}.$$

Therefore

$$\begin{aligned}
\int_0^t \int_0^{2\pi} u^2(1+t-r)^{\alpha-2} r dr d\theta &\leq \frac{1}{1-\alpha} \int_0^t \int_0^{2\pi} u^2 \partial_r \left(r(1+t-r)^{\alpha-1} \right) dr d\theta \\
&\leq \frac{1}{1-\alpha} \left(\int_0^t \int_0^{2\pi} - \left(\partial_r u^2 \right) (1+t-r)^{\alpha-1} r dr d\theta + \left[\int_0^{2\pi} u^2 (1+t-r)^{-\rho} r \right]_0^t \right) \\
&\leq \frac{1}{1-\alpha} \left(2 \int_0^t \int_0^{2\pi} |u \partial_r u| (1+t-r)^{\alpha-1} r dr d\theta + t \int_0^{2\pi} u^2(t,\theta) d\theta \right).
\end{aligned}$$

We have

$$\begin{aligned}
&\int_0^t \int_0^{2\pi} |u \partial_r u| (1+t-r)^{\alpha-1} r dr d\theta \\
\leq &\left(\int_0^t \int_0^{2\pi} u^2 (1+t-r)^{\alpha-2} r dr d\theta \right)^{\frac{1}{2}} \left(\int_0^t \int_0^{2\pi} (\partial_r u)^2 (1+t-r)^{\alpha} r dr d\theta \right)^{\frac{1}{2}}
\end{aligned}$$

and

$$\begin{aligned} t\int_0^{2\pi} u^2(t,\theta)d\theta &\leq t\int_t^\infty\int_0^{2\pi} |\partial_r(u^2)|drd\theta \\ &\leq 2t\int_t^\infty\int_0^{2\pi} |u\partial_r u|\frac{(1+t-r)^{\frac{\beta}{2}}}{(1+t-r)^{\frac{\beta}{2}}}\frac{r}{t}drd\theta \\ &\leq 2\left(\int_t^\infty\int_0^{2\pi} u^2(1+r-t)^{-\beta}rdrd\theta\right)^{\frac{1}{2}}\left(\int_t^\infty\int_0^{2\pi} (\partial_r u)^2(1+r-t)^{\beta}rdrd\theta\right)^{\frac{1}{2}}. \end{aligned}$$

Since $\beta > 1$, we have $\beta > 2-\beta$. Thanks to the estimate (3.12.15) in the region $r > t$, we obtain

$$\begin{aligned} &\int_0^t\int_0^{2\pi} u^2(1+t-r)^{\alpha-2}rdrd\theta \\ \leq &C(\rho,\alpha)\left(\int_0^t\int_0^{2\pi}(\partial_r u)^2(1+t-r)^{\alpha}rdrd\theta + \int_t^\infty\int_0^{2\pi}(\partial_r u)^2(1+r-t)^{\beta}rdrd\theta\right) \end{aligned}$$

This concludes the proof of Proposition 3.12.7. □

3.12.5 Weighted Klainerman-Sobolev inequality

Proposition 3.12.8. *We have the inequality*

$$|f(t,x)v^{\frac{1}{2}}(|x|-t)| \lesssim \frac{1}{\sqrt{1+t+|x|}\sqrt{1+||x|-t|}}\sum_{I\leq 2}\|v^{\frac{1}{2}}(.-t)Z^I f\|_{L^2}.$$

Proof. We introduce the decomposition

$$f = f_1 + f_2,$$

where

$$f_1 = \chi\left(\frac{r}{t}\right)f, \quad f_2 = \left(1-\chi\left(\frac{r}{t}\right)\right)f,$$

and χ is a cut-off such that $\chi(\rho) = 1$ for $\rho \leq \frac{1}{2}$ and $\chi(\rho) = 0$ for $\rho \geq \frac{2}{3}$. Since the quantities $Z^I\chi$ are bounded, it is sufficient to prove the proposition for f_1 and f_2.

For f_1, we introduce the function $f_t = f_1(t,tx)$. The Sobolev embedding $H^2 \hookrightarrow L^\infty$ gives

$$\begin{aligned} \|f_t\|_{L^\infty} &\lesssim \sum_{|\alpha|\leq 2}\|\nabla^\alpha f_t\|_{L^2} \\ &\lesssim \frac{1}{t}\sum_{|\alpha|\leq 2}\|t^\alpha\nabla^\alpha f_1\|_{L^2}. \end{aligned}$$

In the region $r \leq \frac{2t}{3}$ we have $-t \leq r-t \leq -\frac{t}{3}$, therefore

$$|t\nabla\varphi| \lesssim |(r-t)\nabla\varphi| \lesssim \sum_{Z\in\mathcal{Z}}|Z\varphi|.$$

Moreover, in this region $v(|x|-t) \sim v(t)$, so

$$\begin{aligned} |f_1(t,x)v^{\frac{1}{2}}(|x|-t)| &\lesssim \frac{1}{t}\sum_{I\leq 2}\|v^{\frac{1}{2}}(t)Z^I f_1\|_{L^2} \\ &\lesssim \frac{1}{\sqrt{1+t+|x|}\sqrt{1+||x|-t|}}\sum_{I\leq 2}\|v^{\frac{1}{2}}(.-t)Z^I f_1\|_{L^2}. \end{aligned}$$

For f_2 we write

$$\begin{aligned}
&(1+t+r)(1+|t-r|)v(r-t)(f_2(t,r,\theta))^2 \\
&\lesssim \int_{\frac{t}{2}}^{r} \partial_\rho \left((1+t+\rho)(1+|t-\rho|)v(\rho-t)f_2(t,\rho,\theta)^2\right) d\rho \\
&\lesssim \sum_{0\leq\alpha\leq 1} \int_{\frac{t}{2}}^{r} \int_0^{2\pi} |\partial_\theta^\alpha \partial_\rho \left((1+t+\rho)(1+|t-\rho|)v(\rho-t)f_2(t,\rho,\theta)^2\right)| d\rho d\theta
\end{aligned}$$

where we have used the Sobolev embedding $W^{1,1}(\mathbb{S}^1) \hookrightarrow L^\infty(\mathbb{S}^1)$. We estimate the terms appearing when we distribute the derivation ∂_ρ from left to right.

$$\begin{aligned}
|(1+|t-\rho|)v(\rho-t)\partial_\theta^\alpha f_2^2| &\lesssim \rho|v(\rho-t)\partial_\theta^\alpha f_2^2|, \\
|(1+t+\rho)v(\rho-t)\partial_\theta^\alpha f_2^2| &\lesssim \rho|v(\rho-t)\partial_\theta^\alpha f_2^2|, \\
|(1+t+\rho)(1+|t-\rho|)v'(\rho-t)\partial_\theta^\alpha f_2^2| &\lesssim \rho|(1+|t-\rho|)v'(\rho-t)||\partial_\theta^\alpha f_2^2| \qquad \lesssim \rho|v(\rho-t)\partial_\theta^\alpha f_2^2|, \\
|(1+t+\rho)v(\rho-t)(1+|t-\rho|)\partial_\rho\partial_\theta^\alpha f_2^2| &\lesssim \rho|v(\rho-t)|\sum_{Z\in\mathcal{Z}} |Z\partial_\theta^\alpha f_2^2|,
\end{aligned}$$

where we have used in the third inequality $|sv'(s)| \leq v(s)$. Therefore

$$|(1+t+r)(1+|t-r|)v(r-t)(f_2(t,r,\theta))^2| \lesssim \sum_{0\leq\alpha\leq 1}\sum_{Z\in\mathcal{Z}} \|v^{\frac{1}{2}}\partial_\theta^\alpha Z f_2\|_{L^2}^2 \lesssim \sum_{I\leq 2} \|v^{\frac{1}{2}} Z^I f_2\|_{L^2}^2.$$

This concludes the proof of Proposition 3.12.8.

$\square$

Chapitre 4

Non CMC solutions to the constraint equations in the compact hyperbolic case

This chapter is a joint work with Romain Gicquaud. It is independant from Chapters 2 and 3. We give it here in its original form [22].

4.1 Introduction

General relativity describes the universe as a (3+1)-dimensional manifold $\mathcal{M}$ endowed with a Lorentzian metric $\mathbf{g}$. The Einstein equations describe how non-gravitational fields influence the curvature of $\mathbf{g}$:

$$\mathbf{Ric}_{\mu\nu} - \frac{\mathbf{Scal}}{2}\mathbf{g}_{\mu\nu} = 8\pi \mathbf{T}_{\mu\nu},$$

where $\mathbf{Ric}$ and $\mathbf{Scal}$ are respectively the Ricci tensor and the scalar curvature of the metric $\mathbf{g}$ and $\mathbf{T}_{\mu\nu}$ is the sum of the energy-momentum tensors of all the non-gravitational fields.

Einstein equations can be formulated as a Cauchy problem with initial data given by a set $(M, \widehat{g}, \widehat{K})$, where M is a 3-dimensional manifold, $\widehat{g}$ is a Riemannian metric on M and $\widehat{K}$ is a symmetric 2-tensor on M. $\widehat{g}$ and $\widehat{K}$ correspond to the first and second fundamental forms of M seen as an embedded space-like hypersurface in the universe $(\mathcal{M}, \mathbf{g})$ solving the Einstein equations.

It turns out that the Einstein equations imply compatibility conditions on $\widehat{g}$ and $\widehat{K}$ known as the constraint equations:

$$\begin{cases} \mathrm{Scal}_{\widehat{g}} + (\mathrm{tr}_{\widehat{g}}\,\widehat{K})^2 - |\widehat{K}|^2_{\widehat{g}} = 2\rho, & (4.1.1a) \\ \mathrm{div}_{\widehat{g}}\,\widehat{K} - d(\mathrm{tr}_{\widehat{g}}\,\widehat{K}) = j, & (4.1.1b) \end{cases}$$

where, denoting by N the unit future-pointing normal to M in $\mathcal{M}$, one has

$$\rho = 8\pi \mathbf{T}_{\mu\nu} N^\mu N^\nu, \quad j_i = 8\pi \mathbf{T}_{i\mu} N^\mu.$$

We assume here that μ and ν go from 0 to 3 and denote spacetime coordinates while Latin indices go from 1 to 3 and correspond to coordinates on M.

In this article, to keep things simple, we will consider no field but the gravitational one (vacuum case). As a consequence, we impose $\mathbf{T} \equiv 0$. We will also assume that the spacetime possesses a $\mathbb{S}^1$-symmetry generated by a spacelike Killing vector field. This allows for a reduction of the $(3+1)$-dimensional study of the Einstein equations to a $(2+1)$-dimensional problem. This symmetry assumption has been introduced and studied by Y. Choquet-Bruhat and V. Moncrief in [11] (see also [8]) in the case of a spacetime of the form $\Sigma \times \mathbb{S}^1 \times \mathbb{R}$, where Σ is a compact 2-dimensional manifold of genus $G \geq 2$, $\mathbb{S}^1$ corresponds to the orbit of the $\mathbb{S}^1$-action and $\mathbb{R}$ is the time axis. They proved the existence of global solutions corresponding to perturbations of a particular expanding spacetime. In [11], they use solutions of the constraint equations with constant mean curvature (CMC, i.e. constant $\mathrm{tr}_{\widehat{g}} \widehat{K}$) on the spacelike hypersurface $\Sigma \times \mathbb{S}^1 \times \{0\}$ as initial data. The construction of such solutions is fairly direct. In this article we shall generalize their construction to more general initial data allowing for non-constant mean curvature.

The method which is generally used to construct initial data for the Einstein equations is the conformal method which consists in decomposing the metric $\widehat{g}$ and the second fundamental form $\widehat{K}$ into given data and unknowns that have to be adjusted so that $\widehat{g}$ and $\widehat{K}$ solve the constraint equations, see Section 4.2. The equations for the unknowns, namely a positive function playing the role of a conformal factor and a 1-form, are usually called the conformal constraint equations.

These equations have been extensively studied in the case of constant mean curvature (CMC) since the system greatly simplifies in this case. We refer the reader to the excellent review article [4] for an overview of known results in this particular case. The non-CMC case remained open for a couple of decades. Only the case of nearly constant mean curvature was studied. Two major breakthroughs were obtained in [27], [42] and [16] concerning the far from CMC case. A comparison of these methods is given in [23].

In this article, we follow the method described in [16]. Namely, we give the following criterion: if a certain limit equation admits no non-zero solution, the conformal constraint equations admit at least one solution.

This approach has been generalized to the asymptotically hyperbolic case in [24] and to the asymptotically cylindrical case in [18]. The asymptotically Euclidean case [17] and the case of compact manifolds with boundary [21] are currently work in progress since new ideas have to be found to get the criterion.

The outline of the paper is as follows. In Section 4.2, we show how the Einstein equations reduce to a $(2+1)$-dimensional problem in the case of a $\mathbb{S}^1$-symmetry and exhibit the analog of the conformal constraint equations in this case. We also state Theorem 4.2.1 which is the main result of this article and Corollary 4.2.3 which gives an example of application of Theorem 4.2.1. Section 4.3 is devoted to the proof of Theorem 4.2.1. Finally, Section 4.4 contains the proof of Corollary 4.2.3.

4.2 Preliminaries

4.2.1 Reduction of the Einstein equations

Before discussing the constraint equations, we briefly recall the form of the Einstein equations in the presence of a spacelike translational Killing vector field. We follow here the exposition in [8, Section XVI.3].

We recall that we want to write the Einstein equations on the manifold $\mathcal{M} = \Sigma \times \mathbb{S}^1 \times \mathbb{R}$, where Σ is a Riemannian surface and $\mathbb{R}$ denotes the time direction, for some metric $\mathbf{g}$ which is invariant under translation along the $\mathbb{S}^1$-direction. We let x^3 denote the coordinate along the $\mathbb{S}^1$- direction (seen as $\mathbb{R}/\mathbb{Z}$), choose local coordinates x^1, x^2 on Σ and denote by x^0 the time coordinate.

A metric $\mathbf{g}$ on $\mathcal{M}$ admitting ∂_3 as a Killing vector field has the form

$$\mathbf{g} = \widetilde{g} + e^{2\gamma}\left(dx^3 + A\right)^2,$$

where $\widetilde{g}$ is a Lorentzian metric on $\Sigma \times \mathbb{R}$, A is a 1-form on $\Sigma \times \mathbb{R}$ and γ is a function on $\Sigma \times \mathbb{R}$. Since ∂_3 is a Killing vector field, $\widetilde{g}$, A and γ do not depend on x^3. We set $F = dA$ the field strength of A. The Ricci tensor $\mathbf{Ric}$ of $\mathbf{g}$ can be computed in terms of $\widetilde{g}$, A and γ. In the basis $(dx^0, dx^1, dx^2, dx^3 + A)$, the vacuum Einstein equations ($\mathbf{Ric} = 0$) become

$$\begin{cases} 0 = \mathbf{Ric}_{\alpha\beta} = \widetilde{\mathrm{Ric}}_{\alpha\beta} - \frac{1}{2}e^{2\gamma}F_{\alpha}{}^{\lambda}F_{\beta\lambda} - \widetilde{\nabla}^2_{\alpha,\beta}\gamma - \nabla_\alpha\gamma\nabla_\beta\gamma, & (4.2.1a) \\ 0 = \mathbf{Ric}_{\alpha 3} = \frac{1}{2}e^{-\gamma}\widetilde{\nabla}_\beta\left(e^{3\gamma}F_\alpha{}^\beta\right), & (4.2.1b) \\ 0 = \mathbf{Ric}_{33} = -e^{-2\gamma}\left(-\frac{1}{4}e^{2\gamma}F_{\alpha\beta}F^{\alpha\beta} + \widetilde{g}^{\alpha\beta}\nabla_\alpha\gamma\nabla_\beta\gamma + \widetilde{g}^{\alpha\beta}\widetilde{\nabla}^2_{\alpha,\beta}\gamma\right), & (4.2.1c) \end{cases}$$

where the indices α, β and λ go from 0 to 2, and the indices are raised with respect to the metric $\widetilde{g}$. The equation (4.2.1b) is equivalent to $d(*e^{3\gamma}F) = 0$. So we are going to assume that $*e^{3\gamma}F$ is an exact 1-form. Therefore, there exists a potential $\omega : \Sigma \times \mathbb{R} \to \mathbb{R}$ such that $e^{3\gamma}F = d\omega$.

Defining $\overline{g} = e^{2\gamma}\widetilde{g}$, we obtain the following system for $\overline{g}$, γ and ω:

$$\begin{cases} \Box_{\overline{g}}\omega - 4\overline{\nabla}^\alpha\gamma\overline{\nabla}_\alpha\omega = 0, & (4.2.2a) \\ \Box_{\overline{g}}\gamma - \frac{1}{2}e^{-4\gamma}\overline{\nabla}^\alpha\omega\overline{\nabla}_\alpha\omega = 0, & (4.2.2b) \\ \overline{\mathrm{Ric}}_{\alpha\beta} - 2\overline{\nabla}_\alpha\gamma\overline{\nabla}_\beta\gamma - \frac{1}{2}e^{-4\gamma}\overline{\nabla}_\alpha\omega\overline{\nabla}_\beta\omega = 0, & (4.2.2c) \end{cases}$$

where $\Box_{\overline{g}} = \overline{g}^{\alpha\beta}\overline{\nabla}^2_{\alpha,\beta}$ is the d'Alembertian associated to the metric $\overline{g}$, $\overline{\mathrm{Ric}}$ is its Ricci tensor and the indices are raised with respect to $\overline{g}$. We introduce the following notation

$$u := (\gamma, \omega),$$

together with the scalar product

$$\partial_\alpha u \cdot \partial_\beta u := 2\partial_\alpha\gamma\partial_\beta\gamma + \frac{1}{2}e^{-4\gamma}\partial_\alpha\omega\partial_\beta\omega.$$

We are going to consider the Cauchy problem for the system (4.2.2). As for the general Einstein equations, the initial data for this system have to satisfy some constraint equations.

4.2.2 The constraint equations

We write the metric $\overline{g}$ under the following form:

$$\overline{g} = -N^2dt^2 + g_{ij}\left(dx^i + \beta^i dt\right)\left(dx^i + \beta^j dt\right)$$

The coefficient N is called the lapse, while the vector β is called the shift. g is the Riemannian metric induced by $\overline{g}$ on the slices of constant t. We consider the initial data for the spacelike surface Σ which is the constant $t = 0$ hypersurface of $\Sigma \times \mathbb{R}$. We also use the notation

$$\partial_t = \partial_0 - \mathcal{L}_\beta,$$

where $\mathcal{L}_\beta$ is the Lie derivative associated to the vector field β. With this notation, the second fundamental form of $\Sigma \subset \Sigma \times \mathbb{R}$ reads

$$K_{ij} = -\frac{1}{2N}\partial_t g_{ij}.$$

We denote by τ the mean curvature of Σ:

$$\tau := g^{ij}K_{ij}.$$

The constraint equations are obtained by taking the $\partial_t - \partial_t$ and the $\partial_t - \partial_i$ components of the Einstein equations:

$$
\begin{cases}
\overline{\mathrm{Ric}}_{ti} - \frac{\overline{\mathrm{Scal}}}{2}\overline{g}_{ti} = N\left(\partial_i \tau - D^i K_{ij}\right) = \partial_t u \cdot \partial_i u, & (4.2.3a) \\
\overline{\mathrm{Ric}}_{tt} - \frac{\overline{\mathrm{Scal}}}{2}\overline{g}_{tt} = \frac{N^2}{2}\left(\mathrm{Scal} - |K|^2 + \tau^2\right) = \partial_t u \cdot \partial_t u + \frac{N^2}{2}\overline{g}^{\alpha\beta}\partial_\alpha u \cdot \partial_\beta u, & (4.2.3b) \\
 & (4.2.3c)
\end{cases}
$$

where Scal is the scalar curvature of the metric g and D is its Levi-Civita connection. Equation (4.2.3a) is called the *momentum constraint* while Equation (4.2.3b) is known as the *Hamiltonian constraint*.

4.2.3 The conformal method

In order to construct solutions to the system (4.2.3), we are going to use the well-known conformal method which we explain now.

Given a Riemann surface Σ of genus $G \geq 2$, we let g_0 be a metric on Σ with constant scalar curvature $\mathrm{Scal}_0 \equiv -1$ and look for a metric g in the conformal class of g_0:

$$g = e^{2\varphi} g_0$$

for some function $\varphi : \Sigma \to \mathbb{R}$. We also decompose K into a pure trace part and a traceless part,

$$K_{ij} = \frac{\tau}{2} g_{ij} + H_{ij},$$

and, following [11], we set

$$\dot{u} := \frac{e^{2u}}{N}\partial_t u.$$

The system (4.2.3) then becomes

$$
\begin{cases}
\nabla^i H_{ij} = -\dot{u} \cdot \partial_j u + \frac{e^{2\varphi}}{2}\partial_j \tau, & (4.2.4a) \\
\Delta\varphi + e^{-2\varphi}\left(\frac{1}{2}\dot{u}^2 + \frac{1}{2}|H|^2\right) = e^{2\varphi}\frac{\tau^2}{4} - \frac{1}{2}\left(1 + |\nabla u|^2\right), & (4.2.4b)
\end{cases}
$$

where ∇ denotes the Levi-Civita connection of the metric g_0, Δ is the Laplace-Beltrami operator of g_0 and from now on, unless stated otherwise, all norms are taken with respect to the metric g_0.

In order to solve Equation (4.2.4a), we split H according to the York decomposition (see Proposition 4.3.2 for more details):

$$H = \sigma + LW,$$

where σ is a transverse traceless (TT) tensor, i.e. $\mathrm{tr}_{g_0} \sigma \equiv 0$ and $\nabla^i \sigma_{ij} \equiv 0$, and LW denotes the conformal Killing operator acting on a 1-form W:

$$LW_{ij} = \nabla_i W_j + \nabla_j W_i - \nabla^k W_k g_{0ij}.$$

The system (4.2.4) finally becomes

$$
\begin{cases}
-\frac{1}{2}L^* LW = -\dot{u} \cdot du + \frac{e^{2\varphi}}{2} d\tau, & (4.2.5a) \\
\Delta\varphi + e^{-2\varphi}\left(\frac{1}{2}\dot{u}^2 + \frac{1}{2}|\sigma + LW|^2\right) = e^{2\varphi}\frac{\tau^2}{4} - \frac{1}{2}\left(1 + |\nabla u|^2\right), & (4.2.5b)
\end{cases}
$$

where L^* is the formal L^2-adjoint of L:

$$-\frac{1}{2}L^*LW_j = \nabla^i LW_{ij}.$$

The equations of this system are commonly known as the conformal constraint equations. Equation (4.2.5a) is called the *vector equation* and Equation (4.2.5b) is named the *Lichnerowicz equation*.

Given u, $\dot{u}$, τ and σ we are going to construct solutions to the system (4.2.5) for the unknowns φ and W without any smallness assumption on τ. We follow the approach of [16]. The main theorem we prove is the following:

Theorem 4.2.1. *Given $\dot{u} \in L^\infty(\Sigma, \mathbb{R})$, $u \in W^{1,\infty}(\Sigma, \mathbb{R})$ $\tau \in W^{1,p}(\Sigma, \mathbb{R})$ and $\sigma \in W^{1,p}$ a TT-tensor, where $p > 2$, and assuming that τ vanishes nowhere on Σ, then at least one of the following assertions is true:*

1. *The system* (4.2.5) *admits at least one solution $(\varphi, W) \in W^{2,p}(\Sigma, \mathbb{R}) \times W^{2,p}(\Sigma, T^*\Sigma)$,*
2. *There exists a non-trivial solution $V \in W^{2,p}(\Sigma, T^*\Sigma)$ of the following limit equation*

$$-\frac{1}{2}L^*LW = \alpha\frac{\sqrt{2}}{2}\left|LW\right|\frac{d\tau}{|\tau|} \tag{4.2.6}$$

for some $\alpha \in [0, 1]$.

Remark 4.2.2. Since the surface Σ is of genus $G \geq 2$, there is no conformal Killing vector fields on Σ. Therefore $LW \equiv 0$ imply $W \equiv 0$. In particular, there cannot be any non-zero solution to (4.2.6) with $\alpha = 0$, since in this case we would have

$$0 = \int_\Sigma \left\langle W, -\frac{1}{2}L^*LW\right\rangle d\mu^{g_0} = -\frac{1}{2}\int_\Sigma \left|LW\right|^2 d\mu^{g_0},$$

which immediately implies that W is a conformal Killing vector field.

The proof of this theorem is the subject of Section 4.3.

Corollary 4.2.3. *Assume that the mean curvature τ is such that*

$$\left\|\frac{d\tau}{\tau}\right\|_{L^\infty(\Sigma, T^*\Sigma)} < 1$$

then there exists a solution to the conformal constraint equations (4.2.4).

See Section 4.4 for the proof of this corollary.

4.3 Proof of Theorem 4.2.1

Before tackling the full system of equations in SubsectionsecCoupled, we first study the properties of each equation individually, in Subsection 4.3.1 for the vector equation and in Subsection 4.3.2 for the Lichnerowicz equation.

4.3.1 The vector equation

The main result to study Equation (4.2.4a) is the following:

Proposition 4.3.1. *Given a 1-form $Y \in L^p(\Sigma, T^*\Sigma)$, there exists a unique $W \in W^{2,p}(\Sigma, T^*\Sigma)$ such that*

$$-\frac{1}{2}L^*LW = Y.$$

Moreover, W satisfies

$$\|W\|_{W^{2,p}(\Sigma, T^*\Sigma)} \lesssim \|Y\|_{L^p(\Sigma, T^*\Sigma)}.$$

Proof. We can write

$$\begin{aligned}
-\frac{1}{2}L^*LW_j &= \nabla^i\left(\nabla_i W_j + \nabla_j W_i - \nabla^k W_k g_{0ij}\right)\\
&= \Delta W_j + \nabla^i\nabla_j W_i - \nabla_j\nabla^i W_i\\
&= \Delta W_j + \mathrm{Ric}_{ij}W^i\\
-\frac{1}{2}L^*LW_j &= \Delta W_j - \frac{1}{2}W_j, \qquad (4.3.1)
\end{aligned}$$

where we used the fact that in dimension 2, $\mathrm{Ric} = \frac{\mathrm{Scal}}{2}g_{0ij}$. This Bochner formula will be useful in Section 4.4.

On $W^{1,2}(\Sigma, T^*\Sigma)$, we introduce the following bilinear form

$$a(V,W) := \int_\Sigma \langle LV, LW\rangle\, d\mu^{g_0}.$$

We have

$$\begin{aligned}
a(V,W) &= \int_\Sigma \langle V, L^*LW\rangle\, d\mu^{g_0}\\
&= -2\int_\Sigma \left\langle V, \Delta W - \frac{1}{2}W\right\rangle d\mu^{g_0}\\
&= \int_\Sigma \left(2\langle\nabla V,\nabla W\rangle + \langle V,W\rangle\right) d\mu^{g_0}
\end{aligned}$$

It follows immediately that the bilinear form a satisfies the assumptions of the Lax-Milgram theorem: it is continuous and coercive. So given $Y \in L^p(\Sigma, T^*\Sigma) \subset \left(W^{1,2}(\Sigma, T^*\Sigma)\right)^*$ there exists a unique $W \in W^{1,2}(\Sigma, T^*\Sigma)$ such that $-\frac{1}{2}L^*LW = Y$. It follows from elliptic regularity that $W \in W^{2,p}(\Sigma, T^*\Sigma)$ and that $\|W\|_{W^{2,p}(\Sigma,T^*\Sigma)} \lesssim \|Y\|_{L^p(\Sigma,T^*\Sigma)}$. □

In particular, we get the following result:

Proposition 4.3.2. *Given a symmetric traceless tensor $H \in W^{1,p}$, there exist a unique TT-tensor σ and a unique 1-form W such that*

$$H = \sigma + LW.$$

Proof. From the previous proposition, there exists a unique solution $W \in W^{2,p}$ of

$$-\frac{1}{2}L^*LW = \mathrm{div}_{g_0} H.$$

Setting $\sigma = H - LW$, we have

$$\mathrm{div}_{g_0}\sigma = \mathrm{div}_{g_0} H - \mathrm{div}_{g_0} LW = \mathrm{div}_{g_0} H + \frac{1}{2}L^*LW = 0.$$

Therefore, σ is a TT-tensor. □

4.3.2 The Lichnerowicz equation

The aim of this section is to prove the following proposition :

Proposition 4.3.3. *Let $\dot{u}$, u and τ be given as in Theorem 4.2.1. For any given symmetric traceless 2-tensor $H \in L^\infty$, there exists a unique positive function $\varphi \in W^{2,p}(\Sigma,\mathbb{R})$ solving Equation (4.2.4b). Further φ depends continuously on $H \in L^\infty$ and is bounded from below by a positive constant φ_0 which is independent of H.*

Before proving the proposition, we need to recall a general lemma on semilinear elliptic equations. This is a simple version of the so-called sub and super-solution method we took from [46, Chapter 14].

Lemma 4.3.4. *Given an open interval $I \subset \mathbb{R}$, we consider the following equation for φ on Σ:*

$$\Delta\varphi = f(x, \varphi, \lambda), \tag{4.3.2}$$

where $\lambda \in \Lambda$ is a parameter belonging to Λ, an open subset of Banach space, and f is a function belonging to $L^\infty(\Sigma, \mathbb{R}) \otimes C^1(I \times \Lambda, \mathbb{R})$. We assume further that

- $\frac{\partial f}{\partial \varphi} > 0$,
- *there exist constants $a_0, a_1 \in I$ (that may depend continuously on λ), $a_0 \leq a_1$, such that, for all $x \in \Sigma$, $f(x, a_0, \lambda) < 0$ and $f(x, a_1, \lambda) > 0$.*

Then the equation (4.3.2) *admits a unique solution $\varphi \in W^{2,p}(\Sigma, \mathbb{R})$, $2 < p < \infty$, for all $\lambda \in \Lambda$. Further, φ depends continuously on λ.*

Proof. We first prove the existence of a solution for all $\lambda \in \Lambda$. We denote by Ω the closed subset of $L^\infty(M, \mathbb{R})$ defined by

$$\Omega = \{\varphi \in L^\infty(M, \mathbb{R}), a_0 \leq \varphi \leq a_1\}.$$

We choose a constant $A = A(\lambda) > 0$ such that

$$A > \sup_{(x,\varphi)\in\Sigma\times[a_0,a_1]} \frac{\partial f}{\partial \varphi}(x, \varphi, \lambda)$$

and define a map $F : \Omega \to L^\infty(M, \mathbb{R})$ as follows. Given $\varphi_0 \in \Omega$, we define $F(\varphi_0) := \varphi_1$, where $\varphi_1 \in W^{2,p}(\Sigma, \mathbb{R})$ is the (unique) solution to the following linear equation:

$$-\Delta\varphi_1 + A\varphi_1 = A\varphi_0 - f(x, \varphi_0, \lambda).$$

We argue that $\varphi_1 \in \Omega$ as follows. We have

$$\begin{aligned}
-\Delta\varphi_1 + A\varphi_1 &= A\varphi_0(x) - f(x, \varphi_0, \lambda)\\
&= \int_{a_0}^{\varphi_0(x)} \underbrace{\left(A - \frac{\partial f}{\partial \varphi}(x, \varphi, \lambda)\right)}_{>0} d\varphi + Aa_0 - f(x, a_0, \lambda)\\
&\geq Aa_0 - f(x, a_0, \lambda)\\
&\geq Aa_0;\\
-\Delta\left(\varphi_1 - a_0\right) + A\left(\varphi_1(x) - a_0\right) &\geq 0.
\end{aligned}$$

We set $(\varphi_1 - a_0)_- := \min\{0, \varphi_1 - a_0\}$. Multiplying the previous inequality by $(\varphi_1 - a_0)_-$ and integrating over Σ, we get

$$\begin{aligned}
\int_\Sigma \left[-(\varphi_1 - a_0)_- \Delta\left(\varphi_1 - a_0\right) + A\left(\varphi_1(x) - a_0\right)_-^2\right] d\mu^g &\leq 0\\
\int_\Sigma \left[\left|\nabla(\varphi_1 - a_0)_-\right|^2 + A\left(\varphi_1(x) - a_0\right)_-^2\right] d\mu^g &\leq 0,
\end{aligned}$$

from which we immediately conclude that $(\varphi_1(x) - a_0)_- \equiv 0$, i.e. that $\varphi_1 \geq a_0$. A similar argument proves that $\varphi_1 \leq a_1$. Hence F maps Ω into itself.

We note that for fixed λ, F maps Ω into a bounded subset of $W^{2,p}$. This comes from the fact that $\Sigma \times [a_0, a_1]$ is a compact set over which $f(\cdot, \cdot, \lambda)$ is continuous so $f(x, \varphi, \lambda)$ is bounded independently of $\varphi \in \Omega$ and $x \in \Sigma$. Hence, by elliptic regularity

$$\begin{aligned}\|F(\varphi)\|_{W^{2,p}(\Sigma,\mathbb{R})} &\lesssim \|f(x, \varphi, \lambda)\|_{L^\infty(\Sigma,\mathbb{R})}\\ &\lesssim 1.\end{aligned}$$

Denoting by Ω' the closure of the convex hull of $F(\Omega)$, it follows from the Rellich theorem that Ω' is a compact convex subset of $L^\infty(\Sigma, \mathbb{R})$. By the Schauder fixed point theorem, F admits a fixed point φ. φ then satisfies

$$\begin{aligned}-\Delta\varphi + A\varphi &= A\varphi - f(x, \varphi, \lambda)\\ \Leftrightarrow \Delta\varphi &= f(x, \varphi, \lambda).\end{aligned}$$

Hence φ is a solution to (4.3.2) and by elliptic regularity, $\varphi \in W^{2,p}(\Sigma, \mathbb{R})$.

We next prove that the solution to (4.3.2) is unique given $\lambda \in \Lambda$. It follows then that $a_0 \leq \varphi \leq a_1$. Assume given φ_1, φ_2 two solutions to (4.3.2). We have

$$\begin{aligned}0 &= -\Delta(\varphi_2 - \varphi_1) + f(x, \varphi_2, \lambda) - f(x, \varphi_1, \lambda)\\ &= -\Delta(\varphi_2 - \varphi_1) + (\varphi_2 - \varphi_1) \underbrace{\int_0^1 \frac{\partial f}{\partial \varphi}(x, \varphi_1 + y(\varphi_2 - \varphi_1))dy}_{>0},\end{aligned}$$

from which we immediately conclude that $\varphi_1 \equiv \varphi_2$.

We follow a similar strategy to prove that φ depends continuously on λ. We fix an arbitrary $\lambda_0 \in \Lambda$. There exists $\alpha > 0$ such that

$$\frac{\partial f}{\partial \varphi}(x, \varphi, \lambda_0) \geq \alpha$$

for all $(x, \varphi) \in \Sigma \times [a_0(\lambda_0), a_1(\lambda_0)]$. There exist an $\eta_0 > 0$ and $a_0', a_1' \in I$ such that $B_{\eta_0}(\lambda_0) \subset \Lambda$, $a_0' \leq a_0(\lambda)$, $a_1' \geq a_1(\lambda)$ for all $\lambda \in B_{\eta_0}(\lambda_0)$ and

$$\frac{\partial f}{\partial \varphi}(x, \varphi, \lambda) > \frac{\alpha}{2}$$

on $\Sigma \times [a_0', a_1'] \times B_{\eta_0}(\lambda_0)$. We denote by φ_0 the solution to (4.3.2) with $\lambda = \lambda_0$.

For any $\epsilon > 0$, there exists $\eta > 0$, $\eta < \eta_0$ such that

$$|f(x, \varphi_0, \lambda_1) - f(x, \varphi_0, \lambda_0)| < \frac{\epsilon\alpha}{2}$$

for all $x \in \Sigma$ and all $\lambda \in B_\eta(\lambda_0)$. We denote by φ_1 the solution to (4.3.2) with $\lambda = \lambda_1$ for an arbitrary $\lambda_1 \in B_\eta(\lambda_0)$:

$$\begin{cases}-\Delta\varphi_0 + f(x, \varphi_0, \lambda_0) = 0\\ -\Delta\varphi_1 + f(x, \varphi_1, \lambda_1) = 0\end{cases}$$

Subtracting both equations, we get

$$\begin{aligned}0 &= -\Delta(\varphi_1 - \varphi_0) + f(x, \varphi_1, \lambda_1) - f(x, \varphi_0, \lambda_0)\\ &= -\Delta(\varphi_1 - \varphi_0) + f(x, \varphi_1, \lambda_1) - f(x, \varphi_0, \lambda_1) + f(x, \varphi_0, \lambda_1) - f(x, \varphi_0, \lambda_0)\\ 0 &= -\Delta(\varphi_1 - \varphi_0) + \int_0^1 \frac{\partial f}{\partial \varphi}(x, \varphi_0 + y(\varphi_1 - \varphi_0), \lambda_1)dy(\varphi_1 - \varphi_0) + f(x, \varphi_0, \lambda_1) - f(x, \varphi_0, \lambda_0). \quad (4.3.3)\end{aligned}$$

From our assumptions, we have

$$\int_0^1 \frac{\partial f}{\partial \varphi}(x, \varphi_0 + y(\varphi_1 - \varphi_0), \lambda_1) dy > \frac{\alpha}{2}.$$

Multiplying Equation (4.3.3) by $(\varphi_1 - \varphi_0 - \epsilon)_+ := \max\{0, \varphi_1 - \varphi_0 - \epsilon\} \geq 0$, and integrating over Σ, we get

$$\begin{aligned}
&\int_\Sigma \left(f(x, \varphi_0, \lambda_0) - f(x, \varphi_0, \lambda_1)\right)(\varphi_1 - \varphi_0 - \epsilon)_+ d\mu^{g_0} \\
&\quad = \int_\Sigma \Big[\langle \nabla(\varphi_1 - \varphi_0 - \epsilon)_+, \nabla(\varphi_1 - \varphi_0 - \epsilon)_+ \rangle \\
&\qquad + \int_0^1 \frac{\partial f}{\partial \varphi}(x, \varphi_0 + y(\varphi_1 - \varphi_0), \lambda_1) dy (\varphi_1 - \varphi_0)(\varphi_1 - \varphi_0 - \epsilon)_+ \Big] d\mu^{g_0}, \\
&\int_\Sigma \frac{\epsilon\alpha}{2}(\varphi_1 - \varphi_0 - \epsilon)_+ d\mu^{g_0} \\
&\quad \geq \int_\Sigma \left[|\nabla(\varphi_1 - \varphi_0 - \epsilon)_+|^2 + \frac{\alpha}{2}(\varphi_1 - \varphi_0)(\varphi_1 - \varphi_0 - \epsilon)_+ \right] d\mu^{g_0} \\
0 &\geq \int_\Sigma \left[|\nabla(\varphi_1 - \varphi_0 - \epsilon)_+|^2 + \frac{\alpha}{2}\left((\varphi_1 - \varphi_0 - \epsilon)_+\right)^2 \right] d\mu^{g_0}
\end{aligned}$$

Hence $\varphi_1 - \varphi_0 \leq \epsilon$. Similarly, $\varphi_1 - \varphi_0 \geq -\epsilon$. This proves that the function Ψ mapping λ to φ solving (4.3.2) is continuous from Λ to $L^\infty(\Sigma, I)$. It then follows at once from elliptic regularity that Ψ is continuous as a mapping from Λ to $W^{2,p}(\Sigma, \mathbb{R})$. □

We refer the reader to [41, Section 6] for much stronger versions of the sub and super-solution method. We can now give the proof of Proposition 4.3.3:

Proof of Proposition 4.3.3. The Lichnerowicz equation (4.2.4b) can be rewritten in the form (4.3.2):

$$\Delta\varphi = \underbrace{-e^{-2\varphi}\left(\frac{1}{2}\dot{u}^2 + \frac{1}{2}|H|^2\right) + e^{2\varphi}\frac{\tau^2}{4} - \frac{1}{2}\left(1 + |\nabla u|^2\right)}_{:=f(x,\varphi)}.$$

Since τ^2 is bounded away from zero, the assumption $\frac{\partial f}{\partial \varphi} > 0$ is readily checked. Choosing $a_0 := -\max \ln|\tau|$, we have

$$e^{2a_0}\frac{\tau^2}{4} \leq \frac{1}{4}.$$

So

$$f(x, a_0) \leq e^{2a_0}\frac{\tau^2}{4} - \frac{1}{2}\left(1 + |\nabla u|^2\right) \leq \frac{1}{4} - \frac{1}{2} \leq -\frac{1}{4}.$$

Since f is increasing with φ, we immediately get that if $\varphi < a_0$, then $f(x, \varphi) < 0$. Let now $a_1 \geq 0$ be such that

$$e^{2a_1}\frac{\min \tau^2}{4} > \frac{1}{2}\left(1 + \|\nabla u\|^2_{L^\infty}\right) + \frac{1}{2}\|\dot{u}\|^2_{L^\infty} + \frac{1}{2}\|H\|^2_{L^\infty}.$$

Using the fact that we choose $a_1 \geq 0$, it is a simple matter to check that

$$f(x, a_1) > 0$$

and hence if $\varphi > a_1$, $f(x, \varphi) > 0$.

As a consequence, the Lichnerowicz equation satisfies the assumptions of Lemma 4.3.4. This completes the proof of Proposition 4.3.3. □

4.3.3 The coupled system

Following [44], we use Schaefer's fixed point theorem to study the coupled system (see [25, Chapter 11]):

Theorem 4.3.5. *Let X be a Banach space and $\Phi : X \to X$ a continuous compact mapping. Assume that the set*

$$F := \{x \in X, \exists \rho \in [0,1], x = \rho\Phi(x)\}$$

is bounded. Then Φ has a fixed point:

$$\exists x \in X, x = \Phi(x),$$

and the set of fixed points is compact.

We choose $X = L^\infty(\Sigma, \mathbb{R})$ as a Banach space and construct the mapping Φ as follows: Given $v \in X$,

- From Proposition 4.3.1 there exists a unique $W := W(v) \in W^{2,p}$ solving

$$-\frac{1}{2}L^* LW = -\dot{u} \cdot du + \frac{v^2}{2} d\tau, \tag{4.3.4}$$

  which is Equation 4.2.5a with $e^\varphi = v$. Further W depends continuously on $v \in L^\infty$ for the $W^{2,p}$-norm.
- $W \in W^{2,p}$ can then be continuously mapped to $H := \sigma + LW \in W^{1,p}$
- and, in turn, H can be compactly embedded into L^∞.
- Proposition 4.3.3 yields a unique $\varphi \in W^{2,p}$ solving the Lichnerowicz equation (4.2.4b) with the H we previously found.

Setting $\Phi(v) := e^\varphi \in L^\infty$, we loop the loop providing a continuous compact map $\Phi : X \to X$. Thus, we are almost under the assumptions of Theorem 4.3.5. All we need to check is that the set F is bounded. This is the content of the next proposition:

Proposition 4.3.6. *Assume that the set*

$$F := \{v \in L^\infty(\Sigma, \mathbb{R}), \exists \rho \in [0,1], v = \rho\Phi(v)\}$$

is unbounded. Then there exists a constant $\rho_0 \in [0,1]$ and a non-zero $W \in W^{2,p}$ such that

$$-\frac{1}{2}L^* LW = \frac{\sqrt{2}}{2}\rho_0 |LW| \frac{d\tau}{|\tau|}.$$

Proof. Assuming that F is unbounded, we can find sequences $(\rho_i)_{i\geq 0}$ and $(v_i)_{i\geq 0}$ such that $0 \leq \rho_i \leq 1$, $v_i = \rho_i\Phi(v_i)$ and $\|v_i\|_{L^\infty} \to \infty$. Setting $\varphi_i = \log(\Phi(v_i))$ (i.e. $v_i = \rho_i e^{\varphi_i}$), and defining W_i as the solution to (4.3.4) with $v \equiv v_i$, we get the following equations:

$$\begin{cases} -\frac{1}{2}L^* LW_i = -\dot{u} \cdot du + \rho_i^2 \frac{e^{2\varphi_i}}{2} d\tau, & (4.3.5a) \\ \Delta\varphi_i + e^{-2\varphi_i}\left(\frac{1}{2}\dot{u}^2 + \frac{1}{2}|\sigma + LW_i|^2\right) = e^{2\varphi_i}\frac{\tau^2}{4} - \frac{1}{2}\left(1 + |\nabla u|^2\right), & (4.3.5b) \end{cases}$$

Following [16, 24, 44], we set $\gamma_i := \|e^{\varphi_i}\|_{L^\infty}$ and we introduce the following rescaled objects:

$$\psi_i := \varphi_i - \log(\gamma_i),\ \widetilde{W}_i := \frac{1}{\gamma_i^2} W_i.$$

Note that since we assumed that $\|v_i\|_{L^\infty} = \rho_i\gamma_i \to \infty$, with $0 \leq \rho_i \leq 1$, we also have that $\gamma_i \to \infty$. We will assume without loss of generality that $\gamma_i \geq 1$. The following equations for ψ_i and $\widetilde{W}_i$ follow from the definition:

$$\begin{cases} -\frac{1}{2}L^*L\widetilde{W}_i = -\frac{1}{\gamma_i^2}\dot{u}\cdot du + \rho_i^2\frac{e^{2\psi_i}}{2}d\tau, & (4.3.6a)\\ \frac{1}{\gamma_i^2}\Delta\psi_i + e^{-2\psi_i}\left(\frac{1}{2\gamma_i^4}\dot{u}^2 + \frac{1}{2}\left|\frac{\sigma}{\gamma_i^2} + L\widetilde{W}_i\right|^2\right) = e^{2\psi_i}\frac{\tau^2}{4} - \frac{1}{2\gamma_i^2}\left(1+|\nabla u|^2\right), & (4.3.6b)\end{cases}$$

It follows from the definition of γ_i that $\left\|e^{\psi_i}\right\|_{L^\infty} = \left\|\frac{1}{\gamma_i}e^{\varphi_i}\right\|_{L^\infty} = 1$. Hence, from Proposition 4.3.1 applied to (4.3.6a), we have

$$\begin{aligned}\left\|\widetilde{W}_i\right\|_{W^{2,p}} &\lesssim \left\|-\frac{1}{\gamma_i^2}\dot{u}\cdot du + \rho_i^2\frac{e^{2\psi_i}}{2}d\tau\right\|_{L^p}\\ &\lesssim \frac{1}{\gamma_i^2}\left\|\dot{u}\cdot du\right\|_{L^p} + \left\|d\tau\right\|_{L^p}\\ &\lesssim 1.\end{aligned}$$

Consequently, $\widetilde{W}_i$ is bounded in $W^{2,p}$. Since the embedding $W^{2,p} \hookrightarrow C^1$ is compact, we can assume, up to extraction, that $\widetilde{W}_i$ converges to some $\widetilde{W}_\infty \in W^{2,p}$ for the C^1-norm. We can also assume that $\rho_i \to \rho_\infty \in [0,1]$. All we need to do is to prove that $e^{2\psi_i}$ converges in L^∞ to $f_\infty := \sqrt{2}\frac{|L\widetilde{W}_\infty|}{|\tau|}$.

Indeed, passing to the limit in Equation (4.3.6a), we get that $\widetilde{W}_\infty$ satisfies

$$\begin{aligned}-\frac{1}{2}L^*L\widetilde{W}_\infty &= \rho_\infty^2\frac{f_\infty}{2}d\tau\\ &= \frac{\sqrt{2}}{2}\rho_\infty^2\left|L\widetilde{W}_\infty\right|\frac{d\tau}{|\tau|}.\end{aligned} \tag{4.3.7}$$

Hence, $\widetilde{W}_\infty$ satisfies the limit equation with $\alpha = \rho_\infty^2$. Since $e^{2\psi_i}$ has L^∞-norm 1 and converges in L^∞ to f_∞, we have $\|f_\infty\|_{L^\infty} = 1$. In particular, $L\widetilde{W}_\infty \not\equiv 0$ which proves that $\widetilde{W}_\infty \not\equiv 0$.

To prove convergence of $e^{2\psi_i}$ to f_∞, we show that for any $\epsilon > 0$, there exists an i_0 such that

$$\left|e^{2\psi_i} - f_\infty\right| \leq \epsilon$$

for all $i \geq i_0$. We do it in two steps:

- We first show the upper bound
$$e^{2\psi_i} \leq f_\infty + \epsilon$$
by selecting a smooth function f_+ such that
$$f_\infty + \frac{\epsilon}{2} \leq f_+ \leq f_\infty + \epsilon$$
and proving that for i_0 large enough, $\psi_+ := \frac{1}{2}\log(f_+)$ is a super-solution to (4.3.6b):

$$\frac{1}{\gamma_i^2}\Delta\psi_+ + e^{-2\psi_+}\left(\frac{1}{2\gamma_i^4}\dot{u}^2 + \frac{1}{2}\left|\frac{\sigma}{\gamma_i^2} + L\widetilde{W}_i\right|^2\right) \leq e^{2\psi_+}\frac{\tau^2}{4} - \frac{1}{2\gamma_i^2}\left(1+|\nabla u|^2\right). \tag{4.3.8}$$

Since $f_\infty \geq 0$, $f_+ \geq \frac{\epsilon}{2}$ so ψ_+ is a smooth function. In particular, $|\Delta\psi_+|$ is bounded. Moreover, since $\widetilde{W}_i \to \widetilde{W}_\infty$ in C^1 and $\gamma_i \to \infty$, we have

$$\left|\frac{\sigma}{\gamma_i^2} + L\widetilde{W}_i\right|^2 \to \left|L\widetilde{W}_\infty\right|^2$$

as i tends to infinity. So the condition (4.3.8) can be rephrased as

$$o(1) + \frac{1}{2}\left|L\widetilde{W}_\infty\right|^2 - \frac{\tau^2}{4}f_+^2 \leq 0,$$

where $o(1)$ denotes a sequence of functions tending uniformly to 0 when $i \to \infty$. We have

$$f_+^2 \geq \left(f_\infty + \frac{\epsilon}{2}\right)^2 \geq f_\infty^2 + \frac{\epsilon^2}{4}.$$

This yields, for i big enough,

$$o(1) + \frac{1}{2}\left|L\widetilde{W}_\infty\right|^2 - \frac{\tau^2}{4}f_+^2 \leq o(1) + \frac{\tau^2}{4}f_\infty^2 - \frac{\tau^2}{4}\left(f_\infty^2 + \frac{\epsilon^2}{4}\right) \leq o(1) - \frac{\tau_0^2\epsilon^2}{4} \leq 0,$$

where $\tau_0^2 := \inf_\Sigma \tau^2$ is positive by assumption. Therefore ψ_+ is a super-solution to Equation (4.3.6b) and we obtain, for i big enough

$$\begin{aligned}
\frac{1}{\gamma_i^2}\Delta(\psi_+ - \psi_i) &\leq -\left(e^{-2\psi_+} - e^{-2\psi_i}\right)\left(\frac{\dot{u}^2}{2\gamma_i^4} + \frac{1}{2}\left|L\widetilde{W}_i + \frac{\sigma}{\gamma_i^2}\right|^2\right) + \frac{\tau^2}{4}\left(e^{2\psi_+} - e^{2\psi_i}\right)\\
&\leq \frac{\tau^2}{2}e^{2\psi_i}(\psi_+ - \psi_i)\int_0^1 e^{2\lambda(\psi_+ - \psi_i)}d\lambda\\
&\quad + \left(\frac{\dot{u}^2}{2\gamma_i^4} + \frac{1}{2}\left|L\widetilde{W}_i + \frac{\sigma}{\gamma_i^2}\right|^2\right)e^{-2\psi_i}(\psi_+ - \psi_i)\int_0^1 e^{-2\lambda(\psi_+ - \psi_i)}d\lambda\\
&\leq \underbrace{\left[\frac{\tau^2}{2}e^{2\psi_i}\int_0^1 e^{2\lambda(\psi_+ - \psi_i)}d\lambda + \left(\frac{\dot{u}^2}{2\gamma_i^4} + \frac{1}{2}\left|L\widetilde{W}_i + \frac{\sigma}{\gamma_i^2}\right|^2\right)e^{-2\psi_i}\int_0^1 e^{-2\lambda(\psi_+ - \psi_i)}d\lambda\right]}_{>0}(\psi_+ - \psi_i).
\end{aligned}$$

The maximum principle implies that $\psi_i \leq \psi_+$, for i big enough, so

$$e^{2\psi_i} \leq f_\infty + \epsilon.$$

- Second, we show the lower bound

$$e^{2\psi_i} \geq f_\infty - \epsilon$$

We have to be more careful than for the super-solution, since f_∞ can vanish somewhere. Let f_- be a smooth function such that

$$\sqrt{\max(f_\infty^2 - \epsilon, 0)} \leq f_- \leq \sqrt{\max(f_\infty^2 - \frac{\epsilon}{2}, 0)}.$$

We will work on the open domain $\mathcal{A}$ defined by

$$\mathcal{A} = \{x \in \Sigma, f_-(x) > 0\}.$$

On $\mathcal{A}$, we can define $\psi_- = \frac{1}{2}\ln(f_-)$. We want to show that the following inequality is satisfied on $\mathcal{A}$:

$$\frac{1}{\gamma_i^2}\Delta\psi_- + e^{-2\psi_-}\left(\frac{1}{2\gamma_i^4}\dot{u}^2 + \frac{1}{2}\left|\frac{\sigma}{\gamma_i^2} + L\widetilde{W}_i\right|^2\right) \geq e^{2\psi_-}\frac{\tau^2}{4} - \frac{1}{2\gamma_i^2}(1+|\nabla u|^2). \tag{4.3.9}$$

Since $e^{2\psi_-} > 0$ on $\mathcal{A}$, that is equivalent to showing

$$\frac{1}{\gamma_i^2}e^{2\psi_-}\left(\Delta\psi_- + \frac{1}{2}(1+|\nabla u|^2)\right) + \left(\frac{1}{2\gamma_i^4}\dot{u}^2 + \frac{1}{2}\left|\frac{\sigma}{\gamma_i^2} + L\widetilde{W}_i\right|^2\right) - e^{4\psi_-}\frac{\tau^2}{4} \geq 0.$$

We calculate

$$e^{2\psi_-}\Delta\psi_- = \frac{1}{2}\left[\Delta f_- - \frac{|\nabla f_-|^2}{f_-}\right].$$

We can assume that $\partial\mathcal{A}$ is the disjoint union of smooth curves and denote by r the signed distance function to $\partial\mathcal{A}$ which is positive where $f_\infty \geq \epsilon$. We choose f_- such that $f_- \equiv 0$ whenever $r \leq 0$ and $f_- \equiv \epsilon e^{-1/r}$ if $r > 0$ is sufficiently small for some positive ϵ. For such a choice of f_-, $e^{2\psi_-}\Delta\psi_-$ is bounded on $\mathcal{A}$.

Therefore, as for the upper bound, the condition (4.3.9) can be written

$$o(1) + \frac{1}{2}|LW_\infty|^2 - e^{4\psi_-}\frac{\tau^2}{4} \geq 0.$$

On $\mathcal{A}$ we have $e^{4\psi_-} \leq f_-^2 \leq f_\infty^2 - \frac{\epsilon}{2}$. This yields for i big enough

$$o(1) + \frac{1}{2}|LW_\infty|^2 - e^{4\psi_-}\frac{\tau^2}{4} \geq o(1) + \frac{\tau^2}{4}f_\infty^2 - \frac{\tau^2}{4}\left(f_\infty^2 - \frac{\epsilon}{2}\right) \geq o(1) + \frac{\tau^2}{4}\frac{\epsilon}{2} \geq 0.$$

Since $\psi_-(x) - \psi_i(x) \to -\infty$ when $x \to \partial\mathcal{A}$, $\psi_-(x) - \psi_i(x)$ attains its maximum on $\mathcal{A}$. Therefore, since ψ_- is a subsolution, we can apply the maximum principle on $\mathcal{A}$, to deduce that $\psi_- \leq \psi_i$. This yields

$$\max(f_\infty^2 - \epsilon, 0) \leq e^{4\psi_i}.$$

This concludes the proof of the convergence in L^∞ of $e^{2\psi_i}$ towards f_∞. $\square$

4.4 Proof of Corollary 4.2.3

To prove Corollary 4.2.3, all we need to do is to prove that the limit equation (4.2.6) admits no non-zero solution under the assumption

$$\left\|\frac{d\tau}{\tau}\right\|_{L^\infty(\Sigma, T^*\Sigma)} < 1.$$

We take the scalar product of the limit equation with W and integrate over Σ. From the Bochner formula (4.3.1), we get:

$$\begin{aligned}
-\frac{1}{2}\int_\Sigma |LW|^2\, d\mu^{g_0} &= \alpha\frac{\sqrt{2}}{2}\int_\Sigma |LW|\left\langle W, \frac{d\tau}{|\tau|}\right\rangle d\mu^{g_0}\\
\int_\Sigma |\nabla W|^2\, d\mu^{g_0} + \frac{1}{2}\int_\Sigma |W|^2\, d\mu^{g_0} &\leq \alpha\sqrt{2}\int_\Sigma |\nabla W|\left|\frac{d\tau}{\tau}\right| |W| d\mu^{g_0}\\
&\leq \alpha\int_\Sigma |\nabla W|^2\, d\mu^{g_0} + \frac{\alpha}{2}\int_\Sigma \left|\frac{d\tau}{\tau}\right|^2 |W|^2 d\mu^{g_0}\\
\frac{1}{2}\int_\Sigma |W|^2\, d\mu^{g_0} &\leq \frac{\alpha}{2}\left\|\frac{d\tau}{\tau}\right\|_{L^\infty}^2 \int_\Sigma |W|^2 d\mu^{g_0},
\end{aligned}$$

where we used the well-known inequality $ab \leq \frac{a^2}{2} + \frac{b^2}{2}$ with $a = \sqrt{2}\,|\nabla W|$ and $b = \left|\frac{d\tau}{\tau}\right| |W|$. The last inequality immediately yields that $W \equiv 0$ since we assumed that $\left\|\frac{d\tau}{\tau}\right\|^2_{L^\infty} < 1$ and $\alpha \in [0, 1]$.

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